# THE FROBENIUS EIGENVALUES OF ARTIN-SCHREIER-WITT CURVES ARE GAUSS SUMS

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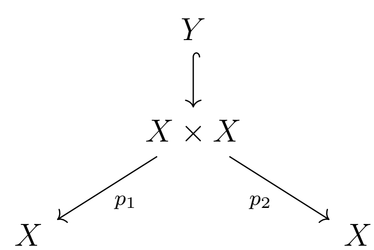
Abstract. We may geometrically diagonalize Frobenius to compute it's trace in order to generalize the Davenport-Hasse theorem and show that eigenvalues of Frobenius of Artin-Schrier-Witt covers are also Gauss sums. We propose this as a method of recovering the slopes of their Jacobians.

## 1. (Brief) Background

I've felt for a long time that automorphisms of curves should control or at least exert serious force on the slopes on their Jacobians. This playful note is toward exploring this force, and outlining a possible method of exploiting it.

Correspondences on $X$ are endomorphisms of $X$. Given $Y \subseteq X \times X$, the associated correspondence gives an endomorphism of $X$ via $p_2(p_1^{-1}(X))$, if the divisor $Y$ we pick doesn't have weird zeros (so that it doesn't just erase all of $X$).




Given a map $X \xrightarrow{\Delta} X \times X$, take the Frobenius map:

$$X \xrightarrow{F} X \times X$$

$$\Delta X \cap FX = \# X(\mathbb{F}_p)$$

If $Y \simeq FX$ is linearly equivalent, then,

$$\#(Y \cap \Delta X) = \# X(\mathbb{F}_q).$$

Inspired by the beautiful proof of the Lefschetz fixed point theorem via correspondences. My hope is to show that for totally wildly ramified $\mathbb{Z}/p^k$ covers the $p$-divisible groups of their Jacobians always have a slope $1/p^k - p^{k-1}$.

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## 2. Reproof of Davenport-Hasse Geometrically Diagonalizing

Let $k$ be a field of characteristic $p$, such that $K \supseteq F_q$. We begin with an Artin-Schreier curve $A$ with affine equation

$$y^p - y = x^{p-1}$$

*Remark.* Coleman considers the more general case of $y^p - y = x^{p^f-1}$, we do not consider $p^f$ here (yet!), as the $x$ terms of the divisor $g_P$ are more complicated for the $p^f$ case and I'm just trying to do a sanity check.

Consider the following homomorphisms:

$$\begin{aligned}\psi : \mathbb{F}_q &\to \mathrm{Aut}(A)\\ a &\mapsto \psi(a)(x,y,z) = (x, y+a)\\ \chi : \mathbb{F}_q^\times &\to \mathrm{Aut}(A)\\ b &\mapsto \chi(b)(x,y,z) = (b^{(q-1)/m}x, y)\end{aligned}$$

**Definition 2.1.** *We consider the multiplication to be within the group structure of* Aut($A$)*, and the addition to be the addition of cycles.*

*We then define the following automorphism,*

$$G := -\sum_{b\in\mathbb{F}_q} \chi(b)\psi(b),$$

*and consider its associated correspondence $G$.*

We also conflate the notation to also call the divisors associated to these correspondences by the same name.

Here is an excellent source on additive and multiplicative characters to get a feel for them if they are new friends for you. `https://kconrad.math.uconn.edu/blurbs/gradnumthy/Gauss-Jacobi-sums.pdf`

**Theorem 2.2.** *(1) Under equivalence of correspondences, $\phi = G$.*

*Proof.* We will construct a principal divisor $(g_P) \in \mathrm{Div}(A)$ and show it agrees with the divisor $(\phi - G)(P - \infty) := \phi(P) - G(P) - \phi(\infty) + G(\infty)$ on all but finitely many $P$. Given a point $P$, a finite point on $A$ over $k$ such that $x(P) \neq 0$, let

$$g_P(Q) := y(Q) - y(P) - \frac{x(Q)}{x(P)}.$$

varying $Q$,

We have reduced the claim to showing

$$(g_P) = (\phi - G)(P - \infty).$$

We will now calculate zeros and poles of $g_P$ and $(\phi - G)(P - \infty)$ to show that they coincide, thus establishing this equivalence.

We mention this for our pole computations. The point at infinity of $A$ is $\infty := [1 : 0 : 0]$, and our uniformizers are as follows.

$$u_P = \begin{cases} y - y(P) & \text{if } y(P) \neq 0 \text{ and } P \neq [1:0:0] \\ x & \text{if } x(P) = 0 \\ \frac{y}{x} & \text{if } P = [1:0:0] \end{cases}$$

This divisor $(g_P)$ has $p$ zeros, it vanishes at

- (A) the $p-1$ points $(\chi(b)\psi(b))(x(P), y(P))$, $b \in \mathbb{F}_q^*$

$$\begin{aligned} g_P(\chi(b)\psi(b)(x(P), y(P))) &= g_P(bx(P), y(P) + b) \\ &= (y(P) + b) - y(P) - \frac{bx(P)}{x(P)} = 0 \end{aligned}$$

- (B) at the point $(x(P)^p, y(P)^p)$ which is distinct from the (A) points if $x(P)^{p-1} \notin \mathbb{F}_p^\times$.

$$\begin{aligned} g_P(\phi(P)) &= g_P(x(P)^p, y(P)^p) \\ &= y(P)^p - y(P) - \frac{x(P)^p}{x(P)} = 0 \end{aligned}$$

My hope is to show that for totally wildly ramified $\mathbb{Z}/p^k$ covers the $p$-divisible groups of their Jacobians always have a slope $1/p^k - p^{k-1}$.

By Lemma 2.3, we have only one pole at $\infty$ of order $p$. Putting it all together, we get:

$$\begin{aligned} \text{div}(g_P) &= \text{div}_0 g_P - \text{div}_\infty g_P \\ &= \big(\sum_{b=1}^{p-1} [\chi(b)\psi(b)(x(P), y(P))] + [(x(P)^p, y(P)^p)]\big) - p[\infty] \end{aligned}$$

By Lemma 2.4, the divisor $(\phi - G)(P - \infty)$ has the same form.

Thus, we have shown that the divisors $(g_P)$ and $(\phi - G)(P - \infty)$ have the same zeros and poles. The divisors being linearly equivalent gives us an equivalence of their associated correspondences.

□

**Lemma 2.3.** *$(g_P)$ has a pole at $\infty$ of order $p$ and no other poles.*

*Proof.* We mention this for our pole computations: The point at infinity is $[1:0:0]$. The uniformizer of thMy hope is to show that for totally wildly ramified $\mathbb{Z}/p^k$ covers the $p$-divisible groups of their Jacobians always have a slope $1/p^k - p^{k-1}$. e curve at infinity is $T := \frac{y}{x}$ and elsewhere is $y$.

At $\infty$, the curve is $Y^p - YZ = Z$ where $[Y : Z] := [\frac{y}{x} : \frac{z}{x}]$, we can nest this to find the order of the uniformizer.

$$\begin{aligned} Z &= Y^p - Y(Y^p - YZ^{p-1})^{p-1} \\ &= f(Y) = \mathcal{O}(Y^p). \end{aligned}$$

Thus, there is a pole at infinity with order $p$. There are no other poles because the curve is totally wildly ramified at infinity (thus there is only one point above the pole infinity in $\mathbb{P}^1$). □

**Lemma 2.4.** *The divisor*

$$(\phi - G)(P - \infty) = \big(\sum_{b=1}^{p-1}[\chi(b)\psi(b)(x(P), y(P))] + [(x(P)^p, y(P)^p)]\big) - p[\infty].$$

*Proof.*
- $G(\infty) = (p-1)[\infty]$. In other words, $G(\infty)$ takes $\infty$ and sends it to $p-1$ translates of itself. Let's see why, for a fixed $b$ $\chi(b)\psi(b)([1:0:0]) = \chi(b)([1:0:0])\phi(b)([1:0:0]) = [b:0:0][1:b:0] = [b:0:0] = [1:0:0]$.
- $G(P) = \sum_{b\in\mathbb{F}_p^\times}(bx(P), y(P)+b)$
- $\phi(P) = (x(P)^p, y(P)^p)$
- $\phi(\infty) = -\infty$

In other words, $G(\infty) - \phi(\infty) = p[\infty]$, and $\phi(P) - G(P) = [(x(P)^p, y(P)^p)] - \sum_{b\in\mathbb{F}_p^\times}[\chi(b)\phi(b)(x(P), y(P))]$. My hope is to show that for totally wildly ramified $\mathbb{Z}/p^k$ covers the $p$-divisible groups of their Jacobians always have a slope $1/p^k - p^{k-1}$.

Thus,

$$\begin{aligned} \phi(P) - \phi(\infty) + G(\infty) - G(P) &= [(x(P)^p, y(P)^p)] + [\infty] \\ &\quad + (p-1)[\infty] - \sum_{b\in\mathbb{F}_p^\times}[\chi(b)\psi(b)(x(P), y(P))] \\ &= \big(\sum_{b=1}^{p-1}[\chi(b)\psi(b)(x(P), y(P))] + [(x(P)^p, y(P)^p)]\big) - p[\infty] \end{aligned}$$

□

**Corollary 2.5.** *The eigenvalues of Frobenius for this Artin-Schreier-Witt curve $A$ are $G_q$.*

*Proof.* Recall that $H^1_{\acute{e}t}(A)$ is the Divisor class group. Given that the Frobenius map and $G_q$ are equivalences of correspondences, they are also equivalences of operators acting on $H^1(A;\mathbb{Q}_\ell)$. The eigenvalues of an operator on a vector space are eigenvalues on the underlying field. □

Something I have yet to understand is how to explicitly evaluate the eigenvalues of the Gauss sum correspondence on the underlying field explicitly.

## 3. Slope Analysis

Let's review what Manin did for the Artin-Schreier curve to see which explicit eigenvalues we *really* get. We may consider $\psi$ to be a multiplicative character, and $\chi$ to be an additive character. Let $t \in \mathbb{F}_p^\times$ and let choose a lift $\tilde{t}$ to $W(k)$.

Let $\zeta$ be a pth root of unity. We define $\phi_i(t) = \zeta^{i\tilde{t}}$ where , and $\chi_j(t) = -\tilde{t}^{-j}$.

**Theorem 3.1.** *(Thm 4.1 Manin's thesis (2)) The eigenvalues of Frobenius on the curve $X : y^p - y = x^{p-1}$ are sums of the following form.*

$$\tau(\psi_i, \chi_j) = \sum_{t \in \mathbb{F}_p^\times} \phi_i(t)\chi_j(t).$$

*where $1 \leq i \leq p-1$ and $1 \leq j \leq p-2$.*

**Corollary 3.2.** *The slopes of the p-divisible group associated to the curve $X$ are $\{1/(p-1), 2/(p-1), ..., (p-2)/(p-1)\}$.*

*Proof.* We use the eigenvalues above. The key observation of Stickelberger is that for $\lambda = 1 - \zeta$,

$$\tau(\phi_i, \chi_j) = -j^{-1}\lambda^j \mod \lambda^{j+1}.$$

Since $v_p(\lambda) = 1/(p-1)$, this means that $v_p(\tau(\phi_i, \chi_j)) = \frac{j}{p-1}$. So we get $(p-1)$ copies of each $1 \leq j \leq (p-2)$. Each of these eigenvalues has multiplicity $p-1$. □

**Conjecture 3.3.** *The slope decomposition of the Dieduonne module associated to the Jacobian of $A_k$ ($\mathbb{Z}/p^k$ totally wildly ramified cover) always has a piece of slope $1/p^{k-1}(p-1)$.*

Some support for the conjecture: I have calculated (thanks to the ATLAS Northwestern super computer) only pieces of the $p = 3$ here $h = p(p-1) = 6$ case in support of this, so at the moment the conjecture is quite flimsy.

- $y^3 - y = 1/x^2, w^3 - w = -x^2y^2 - x^4y$ is the minimal genus ASW curve $X$ – genus 16, with slope decomposition $G_{1/6} \times 2G_{1/3} \times 4G_{1/2} \times 2G_{2/3} \times G_{5/6}$
- $y^3 - y = 1/x^2, w^3 - w = -x^2y^2 - x^4y - y^5 - y^7$ is an Artin-Schreier-Witt curve with genus 29, with slope decomposition $6G_0 \times G_{1/6} \times 2G_{1/3} \times 11G_{1/2} \times 2G_{2/3} \times G_{5/6} \times 6G_1$

## 4. Artin-Schreier-Witt Case for $\mathbb{Z}/p^2$

We just repeat the above but in greater generality. Slay on his part.

**Definition 4.1.** *Let*

$$C(z, w) := \frac{z^p + w^p - (z + w)^p}{p},$$

*then the equations which define our Artin-Schreier-Witt curve $A$.*

$$w^p - w = C(x^{p-1}, y)$$
$$y^p - y = x^{p-1}$$

*This curve is still totally wildly ramified at infinity) and an extension of the curve in the previous section.*

**Definition 4.2.** *Consider the following homomorphisms:*

$$\psi : \mathbb{F}_q \to \mathrm{Aut}(A)$$
$$a \mapsto \psi(a)(x, y, z) = (x, y + a, w - C(a, y))$$
$$\chi : \mathbb{F}_q \to \mathrm{Aut}(A)$$
$$b \mapsto \chi(b)(x, y, z) = (b^{(q-1)/m}x, y, z)$$

*We define addition as*

$$\psi : a + b(x, y + (a + b), w - (C(a, y) + C(b, y)).$$

*We then define the following automorphism,*

$$G := -\sum_{b \in \mathbb{F}_q} \chi(b)\psi(b),$$

*where the multiplication is given within the automorphism group, and the addition is via addition of cycles. We may regard $G$ as a correspondence on the Artin-Schreier-Witt curve $A$.*

*Remark.* Notice that in the above definition, the image of homomorphism $\psi$ is $\mathbb{Z}/p^2$.

**Theorem 4.3.** *Under equivalence of correspondences, $\phi = G$.*

*Proof.* We construct a principal divisor $g_P$ such that it agrees with $(\phi - G)(P - \infty)$ on all but finitely many $P$. Given a point $P$, a finite point on $A$ over $k$ such that $x(P) \neq 0$, let

$$g_P(Q) := w_Q - w_P - C(\frac{x_Q}{x_P}, y_P) \in \mathrm{Div}(A).$$

This divisor $(g_P)$ has a pole at $\infty$ of order $p^2$ and no other poles, $(g_P)$ also has zeros at the following $p^2$ points:

- Let us observe first the zeros of $g_P(Q)$ where $Q := \phi(P) := (x_P^p, y_P^p, w_P^p)$, this is then
$$\begin{aligned} g_P(\phi(P)) &= w_P^p - w_P - C(\frac{x_P^p}{x_P}, y_P) \\ &= w_P^p - w_P - C(x_P^{p-1}, y_P) \\ &= 0 \end{aligned}$$
- We then observe $g_P(\chi(1)\psi(1)) = g_P(x_P, y_P + 1, w_P + C(b, y_P))$,
$$\begin{aligned} g_P(\chi(1)\psi(1)) &= w_P + C(b, y_P) - w_P - C(b\frac{x_P}{x_P}, y_P) \\ &= C(b, y_P) - C(b, y_P) \\ &= 0. \end{aligned}$$

It is sufficient to then calculate zeros and poles of $(\phi - G)(P - \infty)$, and show it agrees with that of $g_P$ shown above. In other words, Conjecture 4.4.

If we do this, it proves the claim and establishes the theorem since we've shown that $(\phi - G)(P - \infty)$ is principal for all by finitely many $P$. ∎

If we could show the only zeros of $g_P$ are the $p^2$ as above, the only zero of $g_P$ is $\infty$ and it has order $p^2$. We'd be golden. More precisely, it's left to show:

**Conjecture 4.4.** *(points of desire)*

$$(\phi - G)(P - \infty) = (x(P)^p, y(P)^p) + \sum_{b \in \mathbb{F}_p^\times} \chi(b)\phi(b)(x(P), y(P)) - p^2[\infty]$$

**Corollary 4.5.** *The eigenvalues of Frobenius for this Artin-Schreier curve $A$ are $G$.*

*Proof.* Recall that $H^1_{ét}(A)$ is the Divisor class group. Given that the Frobenius map and $G_q$ are equivalences of correspondences, they are also equivalences of operators acting on

## References

[1] Robert Coleman, *On the Frobenius Endomorphisms of Fermat and Artin-Schreier Curves.*
[2] Yuri Manin, *The Theory of Formal Groups over Fields of Finite Characteristic.*