HECKE ORBITS AND HOMOTOPY

CATHERINE RAY

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Thanks to Ananth Shankar for answering my questions on this topic. Thanks to Artem Prikhodko for asking me questions which motivated me to write this up. This writeup is meant to give an intuitive introduction to the Hecke Orbit conjecture, as of now this seems to be passed by word of mouth.

1. Hecke Orbit Conjecture in Homotopy Theory

The philosophy behind (Dream 1: Modeling the LT action) is that a proper understanding of finite subgroups of the Morava stabilizer group requires us to geometrically understand curves with 4 properties.

Definition 1. Let a curve C over a ring R (where R is a k-algebra) have h**-splitting** if the formal group law $\widehat{\mathrm{Jac}}(C)$ splits off (or contains) a one-dimensional summand of height h.

Two of these properties necessitate us to understand all classes of curves with h-*splitting* (i.e., living in specific Newton strata[1]) and families of curves with *deformed* h*-splitting*. This is where the following come in:

(1) the Hecke orbit conjecture
(2) generalization of Elkies supersingularity theorem
(3) how the Torelli locus intersects Newton strata. This is done using (n-gonal) cyclic covers of P^1 which are the easiest curves to deform and work with in group cohomology, and applying the Taniyama-Shimura lemma to see which primes your desired fixed Newton polygon works for.

[1](We actually want to consider central leaves, not Newton strata. To compute the Lubin-Tate action, in order to not invert p, we want to work up to geometric isomorphism, not isogeny.

1.1. **Hecke Orbits.** Let $\mathcal{M}$ be a fixed PEL-type shimura variety associated to a reductive group G. For ease of thought, let us conceptualize $\mathcal{M}$ as parameterizing abelian varieties. The modular stacks we want to work with are of PEL-type, and thus, they have a shitton of symmetry (coming from, for example, a bunch of correspondences). These symmetries preserve all p-adic invariants, including the Newton polygon.

Definition 2. Let T be a set of subgroups permuted by G_k the absolute Galois group of a char p field k. Then we may consider all $x, y \in M$ such that the corresponding abelian varieties A_x and A_y have an isogeny over the algebraic closure $\overline{k}$

$$0 \to G \to A_x \xrightarrow{\epsilon} A_y,$$

where $G \in T$. We define $\{G \to A_x \to B\}_T$ to be all isogenies of with kernel $G \in T$ with the *fixed source* A_x, where B ranges through all possible targets in $\mathcal{M}$ of such an isogeny.

This definition gives us a correspondence, which projects $\{G \to A_x \to B\}_T$ to our fixed A_x on one side, and all possible B arising in $\{G \to A_x \to B\}_T$ on the other.

$$\begin{tikzcd}
 & & \mathcal{M} \arrow[ddll, "\alpha"'] \arrow[ddrr, "\beta"] & & \\
 & & \{G \to A_x \to B\}_T \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
\mathcal{M} & A_x & & \{B\} & \mathcal{M}
\end{tikzcd}$$

So, by pushforward and pullback, we get a map $A_x \mapsto \{B\}$. Note that this takes one point to multiple points.

The $\mathcal{M}$ on top will actually be some specific level, which is larger than the levels downstairs. The α_p-correspondences are $G \in T$ such that G is a locally local group-scheme (built out of extensions of α_p),

(and the α_{ℓ^n}-Hecke correspondences are about...I suppose those built out of extensions of α_{ell^n}, not sure, prime-to-p still confuses me ????).

Understanding how this action takes a non-ordinary point and spreads it out into an army of cockroaches is the aim of the Hecke orbit conjecture. We need to introduce leaves to make this precise.

1.2. **Leaves: Foliations of $\mathcal{M}$.** We consider Hecke orbits related to α_{ℓ^n} (i.e., prime-to-p) isogenies, and α_p isogenies. The first moves points in a central leaf, the second moves points in an isogeny leaf. We can see that α_ℓ keeps us in the same central leaf in the following way. Let

$$\alpha_\ell \to A \xrightarrow{\epsilon} A'$$

be an isogeny over $\overline{k}$, then take the p-divisible groups,

$$0 \to A[p^\infty] \xrightarrow{\tilde{\epsilon}} A'[p^\infty].$$

These are now geometrically isomorphic, as the ℓ-torsion kernel is all dead (not detected by p-torsion).

Definition 3. Central leaves are the *geometric *isomorphism* classes of p-divisible groups.*

The Hecke orbit conjecture is about proving that the *Hecke orbit of any point in a central leaf is dense in that leaf*[2]. Now, central leaves are way way more interesting and important for homotopy theory than being Newton strata. This is because they don't invert p, and we are interested in computing p-torsion groups and thus do not want to invert p.

In particular, if we pick an ordinary point $z \in \mathcal{M}$, take the Zariski closure of its orbit under prime-to-p isogenies, $Z := (t_N^* z)^{\text{zar}}$. Let A_z be the abelian variety associated to z, and let $\text{Aut}_{\mathcal{M}}(A_z)$ be the automorphism group preserving the Shimura variety structure (polarization, level structure and suchlike). Let us denote $Z_z^\wedge$ be the formal completion of the variety at the point z, and C be the central leaf containing z. Then, we have an action of $\text{Aut}_{\mathcal{M}}(A_z)$ on

$$Z_z^\wedge \subseteq \mathcal{C}_z^\wedge.$$

$Z_z^\wedge$ is stable under this action (since it is prime-to-p Hecke stable). We then reduce the problem to showing that all formal subschemes which are stable under this action are actually the entire $M_z^\wedge$. This would allow us to conclude that

$$Z_z^\wedge \simeq C_z^\wedge,$$

i.e, the Hecke orbit conjecture is true (for the central leaf C). No known approaches understand how to use that Z is algebraic.

Definition 4. Between any two central leaves in a Newton stratum, we have a correspondence by iterated α_p-isogenies – so we may study orbits by such isogenies, which gives us the notion of an *isogeny leaf.*

I find the isogeny leaf definition rather unenlightening, let's look at an example.

Example 1. Let $E \times E$ be a product of two supersingular elliptic curves (do I need them to be geometrically iso, not sure). Let's define their isomogeny leaf. We have a map

$$F :: k \longmapsto \left\{ G \subset (E \times E)[p] \,\middle|\, \begin{matrix} G \text{ defined over } k, \\ G \text{ is of order } p. \end{matrix} \right\}$$

This is represented by a $\mathbb{P}^1$. This char p version is hella strange (in char 0, its finite). Its truly surprising, and I've been told that this fact is apparently somewhere in Pink but I haven't found it.

Example 2. Turns out central leaves and isogeny leaves are almost transversal, and that every component of a Newton polygon stratum is up to a finite morphism isomorphic with the product of any of the isogeny leaves with a finite cover of any of the

[2]This is immediate for ordinary points, as they are dense in all of $\mathcal{M}$, restated: $A[\xi]$ is open for the ordinary Newton stratum ξ, i.e., any two ordinary curves are geometricall isomorphic.

central leaves. One way to intuitively see that the two kinds of leaves are transverse is again with our favorite example of two supersingular elliptic curves.

All endomorphisms of $E_s \times E_s$ are defined over $\mathbb{F}_{p^2}$. Let G_s denote the p-div group of E_s. Now, $\mathrm{End}(G_s \times G_s)$ is defined over $\mathbb{F}_{p^2}$, and most of the $\mathbb{P}^1$ is defined over bigger fields. So, given $x \in \mathbb{P}^1(\bar{k}) \backslash \mathbb{P}^1(\mathbb{F}_{p^2})$,

$$(G_s \times G_s)/G_x \simeq G_s \times G_s,$$

since the subgroup is not defined, its not in the same central leaf.

finish thinking about this

1.3. ***h*-splitting of AbVar.** Let k be a numberfield. I wanted to mention:
Relation of Intersections of Hecke orbits to isogeny: Let's take y and z to be the points corresponding to the elliptic curves E and E' in $X(1)_k$, and take their Zariski closures Y and Z resp. in $X(1)_{O_k}$. Then, the geometric intersection points of Z and $(t_N)_*Y$ correspond to pairs (finite place p of k, cyclic isogeny of degree N between $E_\kappa(p)$ and $E'_\kappa(p)$ where $\kappa(p)$ is the algebraic closure of the residue field).

Remark. We pullback pushforward to get maps on

$$H^0(A, \Omega^1) \xrightarrow{\alpha^*} H^0(C, \Omega^1) \xrightarrow{\beta_*} H^0(B, \Omega^1),$$

and then use that

$$S_2(\Gamma_0(N)) \simeq H^0(X_0(N), \Omega^1)$$
$$f(z) \mapsto f(z)dz).$$

$$\beta_* \circ \alpha^*$$

acts as a Hecke operator on $S_2(\Gamma_0(N))$ (on the usual prime to p corresp of $C = X_0(pN)$ and $A, B = X_0(N)$). We can see this by carefully tracing through and seeing that we get something which looks like a correspondence which scales lattices (pg 114), i.e., T_p. The prime-to-p case of modular curves is discussed here in more detail than I give (`https://wstein.org/edu/Fall2003/252/lectures/10-31-03/10-31-03.pdf`). I also found this reference, `http://www.digizeitschriften.de/dms/img/?PID=GDZPPN00210749X` which is relevant and richer but with more detail than we need right now.

2. Local Langlands action of J and Hecke Orbits

We see from the above discussion of Hecke correspondences on levels, that a conjugation action on level sets can be thought of as pushing and pulling back through a correspondence.

Question 3. *Why is this an action by a quaternionic division algebra?*

Remark. The action of J in local langlands only appears at infinite level (at the global level). So, how does this global J is relate to the J in the Lubin-Tate action? Its nontrivial, proved by Faltigs originally, and I don't understand it yet: here is a modern reference (Theorem E in `https://arxiv.org/pdf/1211.6357.pdf`).