TOPOLOGY OF ROBOT MOTION PLANNING

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Abstract. In this paper we discuss topological problems inspired by robotics. We study in detail the robot motion planning problem. With any path-connected topological space X we associate a numerical invariant $\mathbf{TC}(X)$ measuring the "*complexity of the problem of navigation in X*." We examine how the number $\mathbf{TC}(X)$ determines the structure of motion planning algorithms, both deterministic and random. We compute the invariant $\mathbf{TC}(X)$ in many interesting examples. In the case of the real projective space $\mathbb{R}\mathbf{P}^n$ (where $n \neq 1, 3, 7$) the number $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) - 1$ equals the minimal dimension of the Euclidean space into which $\mathbb{R}\mathbf{P}^n$ can be immersed.

Key words: Robot motion planning algorithms, navigational complexity of configuration spaces, collision avoiding, cohomological lower bound, immersions of projective spaces

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1. Introduction

This paper represents a slightly extended version of a mini-course consisting of four lectures delivered at the Université de Montréal in June 2004. The main goal was to give an introduction to the topological robotics and in particular to describe a topological approach to the robot motion planning problem. This new theory appears to be useful both in robotics and in topology. In robotics, it explains how the instabilities in the robot motion planning algorithms depend on the homotopy properties of the robot's configuration space. In topology, if one regards the topological spaces as configuration spaces of mechanical systems, one discovers a new interesting homotopy invariant $\mathbf{TC}(X)$ which measures the "*navigational complexity*" of X. The invariant $\mathbf{TC}(X)$ is similar in spirit to the Lusternik – Schnirelmann category $\mathrm{cat}(X)$ although in fact $\mathbf{TC}(X)$ and $\mathrm{cat}(X)$ are independent, as simple examples (given below) show.

The topological approach to the robot motion planning problem was initiated by the author in Farber (2003; 2004). It was inspired by the earlier well-known work of Smale (1987) and Vassil′iev (1988) on the theory of topological complexity of algorithms of solving polynomial equations. The approach of Farber (2003; 2004) was also based on the general theory of robot motion planning algorithms described in the book of J.-C. Latombe (1991). It is my pleasure to acknowledge the importance of discussions of the initial versions of Farber (2003; 2004) with Dan Halperin and Micha Sharir in summer 2001 in Tel Aviv. The theory of a genus of a fiber space developed by Schwarz (1966) plays a technically important role in our approach as well as in the work of Smale and Vassiliev.

Further developments and applications of the theory of topological complexity of robot motion planning of Farber (2003; 2004) are also mentioned in these notes. They include the study of collision free motion planning algorithms in Euclidean spaces (Farber and Yuzvinsky, 2004) and on graphs (Farber, 2005) and also applications to the immersion problem for the real projective spaces (Farber et al., 2003).

These notes also include some new material. We explain how one may construct an explicit motion planning algorithm in the configuration space of n distinct particles in $\mathbb{R}^m$ having topological complexity $\leq n^2$. Such an algorithm may have some practical applications. We also analyze the complexity of controlling many objects simultaneously. Finally we mention some interesting open problems.

The plan of the lectures in Montreal was as follows:

Lecture 1 Introduction. Interesting topological spaces provided by robotics and questions about the standard topological spaces one asks after encounters with robotics.

Lecture 2 The notion of topological complexity of the motion planning problem. The Schwarz genus. Computations of the topological complexity in basic examples.

Lecture 3 Topological complexity of collision free motion planning of many particles in Euclidean spaces and on graphs.

Lecture 4 Motion planning in projective spaces. Relation with the immersion problem for the real projective spaces. Discussion of open problems.

2. First examples of configuration spaces

The ultimate goal of robotics is creating of autonomous robots (Latombe, 1991). Such robots should be able to accept high-level descriptions of tasks and execute them without further human intervention. The input description specifies *what* should be done and the robot decides *how* to do it and performs the task. One expects robots to have sensors and actuators.

A few words about history of robotics. The idea of robots goes back to ancient times. The word *robot* was first used in 1921 by Karel Capek in his play "Possum's Universal Robots." The word *robotics* was coined by Isaac Asimov in 1940 in his book "I, robot."

What is common to robotics and topology? Topology enters robotics through the notion of *configuration space*. Any mechanical system R determines the variety of all its possible states X which is called the configuration space of R. Usually a state of the system is fully determined by finitely many real parameters; in this case the configuration space X can be viewed as a subset of the Euclidean space $\mathbb{R}^k$. Each point of X represents a state of the system and different points represent different states. The configuration spaces X comes with the natural topology (inherited from $\mathbb{R}^k$) which reflects the technical limitations of the system.

Many problems of control theory can be solved knowing only the configuration space of the system. Peculiarities in the behavior of the system can often be explained by topological properties of the system's configuration space. We will discuss this in more detail in the case of the motion planning problem. We will see how *one may predict the character of instabilities of the behavior of the robot knowing the cohomology algebra of its configuration space*.

If the configuration space of the system is known one may often forget about the system and study instead the configuration space viewed with its topology and with some other geometric structures, e.g., with the Riemannian metric.

EXAMPLE 2.1 (Piano movers' problem; Schwartz and Sharir, 1983). In Figure 1 the large rectangles represent the obstacles and the black figures represent different states of the "piano."

We assume that the picture is planar, i.e., the objects move in the horizontal plane only. Of course in practice the obstacles may have much more involved geometry than it is shown on the picture. One has to move the piano from one state to another avoiding the obstacles. The configuration space in this example is 3-dimensional having complicated geometry. The state of the piano is determined by the coordinates of the center and by the orientation.

EXAMPLE 2.2 (The robot arm; Latombe, 1991). Schematically, the robot arm consists of several bars connected by revolving joins (Figure 2). One distinguishes

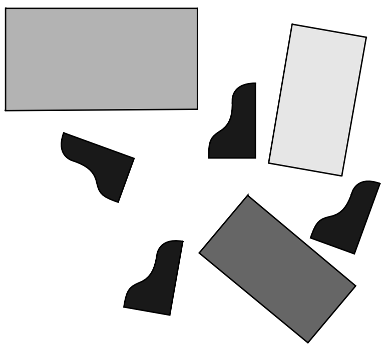

Figure 1.

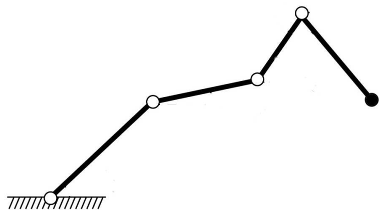

Figure 2.

the spacial case and the planar case (when the bars lie in a single 2-dimensional plane).

The configuration space in this example is

$$X = S^1 \times S^1 \times \cdots \times S^1$$

(the n-dimensional torus) in the planar case and it is

$$X = S^2 \times S^2 \times \cdots \times S^2$$

in the spacial case. We allow the self-intersection of the arm. The space of all configurations of the planar robot arm with no self-intersections is topologically very much different: it is homotopy equivalent to a circle, see the recent work of Connelly et al. (2003).

EXAMPLE 2.3 (The "usual" configuration spaces). Let Y be a topological space and let $X = F(Y, n)$ denote the subset of the Cartesian product $Y \times Y \times \cdots \times Y$ (n times) containing the n-tuples $(y_1, y_2, \ldots, y_n)$ with the property that $y_i \neq y_j$ for $i \neq j$ (Figure 3).

$X = F(Y, n)$ is the configuration space of a system of n particles moving in the space Y avoiding *collisions*. The most interesting special cases are $Y = \mathbb{R}^m$ (the Euclidean space) and when Y is a connected graph.

The configuration spaces $F(\mathbb{R}^m, n)$ were introduced by Fadell and Neuwirth (1962). Nowadays they are standard objects of topology. The configuration spaces $F(\mathbb{R}^2, n)$ and $F(\mathbb{R}^2, n)/\Sigma_n$ appear in the theory of braids. In 1968 V. Arnol′d used information about cohomology of the configuration spaces to study algebraic functions.

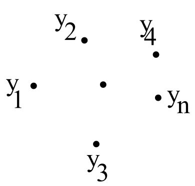

Figure 3.

In robotics it is natural to study the configuration spaces $F(\Gamma, n)$ where Γ is a graph. Such spaces describe several objects moving along a prescribed net Γ (say, the factory floor) avoiding collisions, see Section 27.

3. Varieties of polygonal linkages

In this section we consider the configuration spaces of polygonal linkages. These are remarkable manifolds which describe shapes of closed polygonal chains in robotics; they also appear in many areas of mathematics. The varieties of polygonal linkages carry a set of geometric structures; for example they are Kähler and support several Hamiltonian circle actions. These varieties were studied by Thurston (1987), Walker (1985), Klyachko (1994), Kapovich and Millson (1996), and Hausmann and Knutson (1998). Our exposition mainly follows the work of Klyachko (1994). We describe some basic facts about these varieties referring the reader to the articles mentioned above for more complete information and for proofs.

Fix a vector $a \in \mathbb{R}^m_+$, $a = (a_1, \ldots, a_m)$ consisting of m positive real numbers $a_i > 0$. Define the variety $M(a)$ as follows

$$M(a) = \left\{(z_1, \ldots, z_m); z_i \in S^2, \sum_{i=1}^{m} a_i z_i = 0\right\} \Big/ \mathrm{SO}_3 .$$

Here SO_3 acts diagonally on the product $S^2 \times \ldots \times S^2$. $M(a)$ is the variety of all polygonal shapes in $\mathbb{R}^3$ having the given side lengths (Figure 4).

The first question is whether $M(a)$ in nonempty.

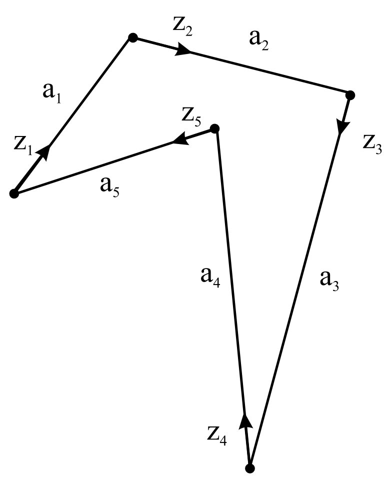

Figure 4.

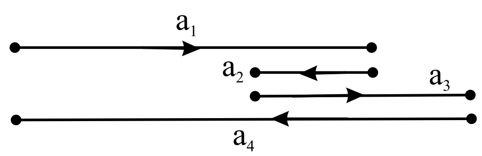

Figure 5.

LEMMA 3.1. *$M(a) \neq \emptyset$ if and only if for any $i = 1, \ldots, m$ one has $a_i \leq |a|/2$, where $|a| = a_1 + a_2 + \cdots + a_m$.*

This follows from the triangle inequality.

DEFINITION 3.2. Vector $a \in \mathbb{R}^m_+$ is *generic* if the equation $\sum_{i=1}^m \epsilon_i a_i = 0$ has no solutions with $\epsilon_i = \pm 1$.

Equivalently, a is generic if the variety $M(a)$ contains no *lined* configurations, i.e., configurations with all the edges lying on a single line (Figure 5).

LEMMA 3.3. *If a is generic then $M(a)$ is a closed smooth manifold of dimension* $2(m-3)$.

3.1. SHORT AND LONG SUBSETS

How does the variety $M(a)$ depend on the vector a? To answer this question we need to introduce the notions of *short* and *long* subsets.

A subset $J \subset \{1, 2, \ldots, m\}$ is called *short* iff

$$\sum_{i \in J} a_i \leq \sum_{j \notin J} a_j$$

One element subsets $J = \{i\}$ are always short assuming that $M(a)$ is nonempty.

$S(a)$ will denote the set of all short subsets. $S(a)$ is a partially ordered set, it is determined by its maximal elements (since a subset of a short subset is short).

EXAMPLE 3.4. Let $a = (1, 1, 1, 2)$. The maximal elements of $S(a)$ are:

$$\{(12), (13), (23), \{4\}\}.$$

LEMMA 3.5. *Assume that the vectors $a, a' \in \mathbb{R}^m_+$ are generic and such that the posets $S(a)$ and $S(a')$ are isomorphic. Then $M(a)$ and $M(a')$ are diffeomorphic.*

See Hausmann and Knutson (1998) for a proof.

3.2. POINCARÉ POLYNOMIAL OF $M(A)$

Klyachko (1994) found a remarkable formula for the Poincaré polynomial of $M(a)$.

THEOREM 3.6. *The Poincaré polynomial of* $M(a)$ *equals*

$$P(t) = \frac{1}{t^2(t^2-1)}\left((1+t^2)^{m-1} - \sum_{J \in S(a)} t^{2|J|}\right) \tag{1}$$

The proof uses the Morse theory, we shall sketch its main points.

Fix a pair of indices $i, j \in \{1, \ldots, m\}$ and consider the smooth function $H\colon M(a) \to \mathbb{R}$ given by

$$H = -\|a_i z_i + a_j z_j\|^2.$$

One may assume without loss of generality that $j = i + 1$. Then H is the negative square of the length of a diagonal of a polygon (Figure 6).

The critical points of H are of types (I), (II), (III) described in Figure 7. In the case of critical points of type (I) one has $z_i = z_j$. For the type (II) $z_i = -z_j$. In the case (III) all the sides of the polygon except z_i and z_j are lined up. The length of the base of the triangle equals

$$\sum_{k \neq i,j} \epsilon_k a_k. \tag{2}$$

Clearly the critical points of types (I) and (II) form critical submanifolds which are diffeomorphic to varieties of polygonal linkages with lower number of edges.

Critical points of type (III) are isolated.

The following Lemma describes the Morse indices:

LEMMA 3.7. *The Morse–Bott index of the critical submanifold* (I) *is* 0. *The Morse–Bott index of the critical submanifold* (II) *is* 2. *The Morse index of any critical point of type* (III) *equals twice the number of minus signs* ϵ_k *appearing in* (2).

We refer to Klyachko (1994) and to Kapovich and Millson (1996) for a proof. We only mention that the first statement concerning the type (I) critical points is

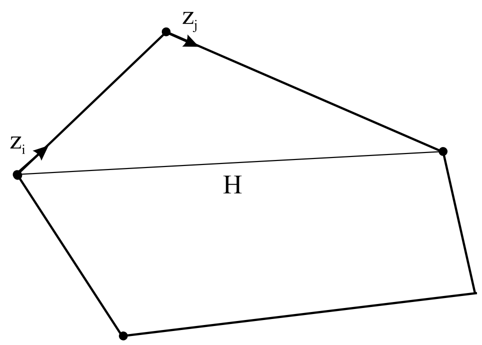

Figure 6.

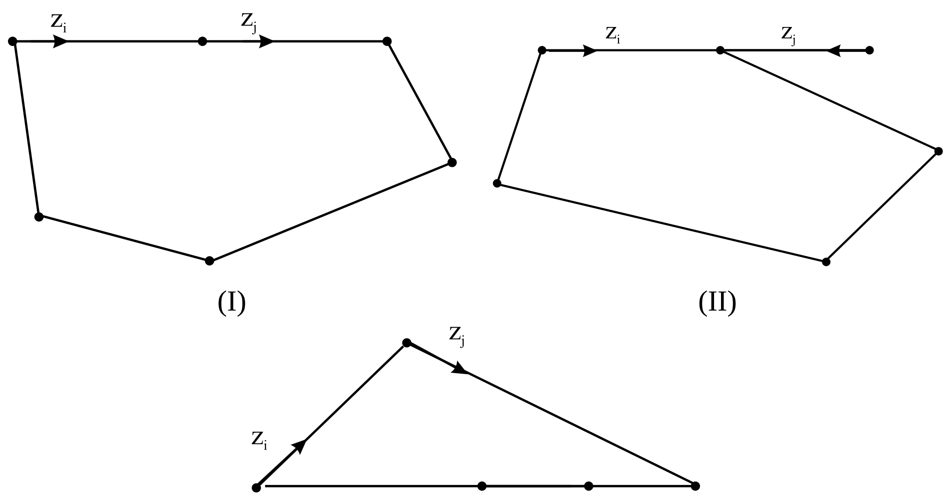

Figure 7.

obvious: the diagonal is then the longest $a_i + a_j$ and therefore the function H has minimum.

Figure 8 shows why each minus sign ϵ_k in the sum (2) gives a two-dimensional family of deformations of the shape of the polygon decreasing (quadratically) the value of the function H.

LEMMA 3.8. *$H: M(a) \to \mathbb{R}$ is a perfect Morse–Bott function.*

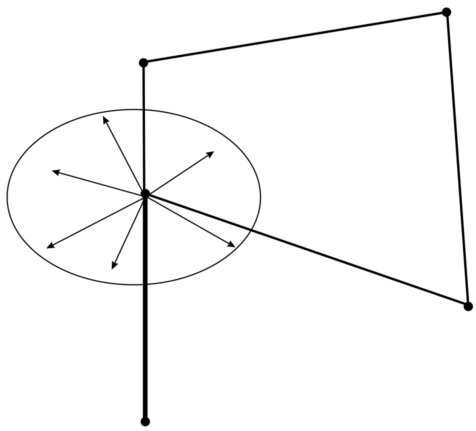

Figure 8.

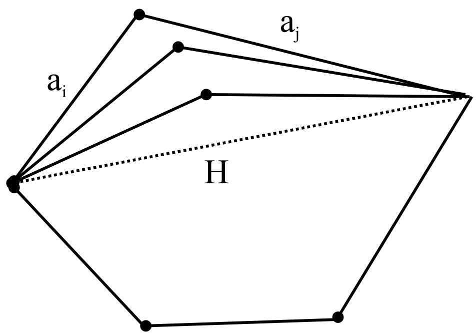

Figure 9.

The proof of this statement uses the existence of a symplectic structure on the manifold $M(a)$ such that the function H generates a Hamiltonian circle action. The latter can be described geometrically as in Figure 9.

One bends the polygon along the specified diagonal. The fixed points of this action are precisely the critical points of H, i.e., the polygons of types (I), (II), (III).

THEOREM 3.9 (Klyachko). *Assume that the vector $a \in \mathbb{R}_+^m$ is generic. Then $M(a)$ admits a symplectic structure such that the circle action described above is Hamiltonian with the function H as the Hamiltonian.*

Perfectness of the function H leads to the following recurrence relation for the Poincaré polynomials. Denote

$$\begin{aligned} a_+ &= (a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_m, a_i + a_j) \in \mathbb{R}_+^{m-1}, \\ a_- &= (a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_m, |a_i - a_j|) \in \mathbb{R}_+^{m-1}. \end{aligned}$$

Here the hat above a symbol means that this symbol should be skipped. We obtain (using the perfectness of H) the following recurrence equation:

$$P_{M(a)}(t) = P_{M(a_+)}(t) + t^2 P_{M(a_-)}(t) + \sum_{|a_i - a_j| < \sum_{k \neq i,j} \epsilon_k a_k < a_i + a_j} t^{2n_\epsilon},$$

where $\epsilon_k = \pm 1$ and n_ϵ is the number of negative ϵ_k. This relation leads eventually to formula (1) for the Poincaré polynomial. The reader may find the continuation of this beautiful story in (Klyachko, 1994).

4. Universality theorems for configuration spaces

How special are configuration spaces of the mechanisms? In other words, we ask if there exist specific topological properties which characterize the configuration spaces among the topological spaces?

Universality theorems for configuration spaces claim (roughly) that all "*reasonable*" topological spaces are configuration spaces of linkages. Many theorems of this type are known. Lebesgue (1950) gave an account of several results including Kempe's universality theorem, not for the configuration space of the mechanism itself but for the orbit of one of its vertices: "*Toute courbe algébrique peut être tracée à l'aide d'un système articulé*."

A theorem of Jordan and Steiner (1999) states:

THEOREM 4.1. *Any compact real algebraic variety $V \subset \mathbb{R}^n$ is homeomorphic to a union of components of the configuration space of a mechanical linkage.*

Kapovich and Millson (2002) prove the following statement:

THEOREM 4.2. *For any smooth compact manifold M there exists a linkage whose moduli space is diffeomorphic to a disjoint union of a number of copies of M.*

Let us explain the terms used here. An *abstract linkage* is a triple

$$\mathcal{L} = (L, \ell, W)$$

where L is a graph, $W \subset V(L)$ is an ordered subset of vertices of L, and $\ell\colon E(L) \to \mathbb{R}_+$ is a function on the set of edges of L. Here W are the *fixed* vertices of L and ℓ is a "*metric*" (length function) on L. A *planar realization* of $\mathcal{L}$ is a map

$$\phi\colon V(L) \to \mathbb{R}^2$$

such that if the vertices $v, w \in V(L)$ are joined by an edge $e \in E(L)$ in L then

$$|\phi(v) - \phi(w)| = \ell(e).$$

Let $W = (v_1, \ldots, v_n)$ and let $Z = (z_1, \ldots, z_n)$ be an ordered set of n points $z_i \in \mathbb{R}^2$. A *planar realization* of $\mathcal{L}$ relative to Z is a realization $\phi\colon V(L) \to \mathbb{R}^2$ as above satisfying an additional requirement that $\phi(v_j) = z_j$ for all $j = 1, \ldots, n$. The set $C(\mathcal{L}, Z)$ of all relative planar realizations of $\mathcal{L}$ is called the *relative configuration space* of $\mathcal{L}$. Elements of $C(\mathcal{L}, Z)$ are all planar realizations of $\mathcal{L}$ such that the vertices of W stay in the prescribed positions Z.

The linkages which we studied in Section 3 are special cases when the graph L is homeomorphic to the circle.

Kapovich and Millson (2002) observe that the configuration space of any planar linkage admits an involution (induced by a reflection of the plane) and this involution is nontrivial if the graph L is connected and the configuration space is not a point. Hence if M^n is a closed manifold $n > 0$ which does not admit a nontrivial involution (such manifolds exist) then M is not homeomorphic to the moduli space of a planar linkage.

5. A remark about configuration spaces in robotics

The notion of configuration space may seem obvious and even trivial for a topologist. But for people in robotics it is not so. In fact in some problems of robotics this notion appears to be even controversial. For a system of great complexity it is unrealistic to assume that its configuration space can be described completely; more reasonably to think that at any particular moment the topology and the geometry of the configuration space are known only partially or approximately.

We want to emphasize that we do not question the existence of the configuration spaces. However in some particular cases it may happen to be too expensive to learn the topology of the configuration space entirely. Then one has to solve the control problems "on-line" and to learn the underlying configuration space at the same time.

It seems plausible that there may exist a better mathematical notion of a configuration space describing a "*partially known*" topological space whose geometry is being gradually revealed.

6. The motion planning problem

In this section we start studying the robot motion planning problem which is the main topic of these lectures.

Imagine that you get into your advanced car and say "*Go home!*" and the car takes you home, automatically, obeying the traffic rules. Such a car must have a GPS (finding its current location) and a computer program suggesting a specific route from any initial state to any desired state. Computer programs of this kind are based on *motion planning algorithms*. In general, given a mechanical system, a motion planning algorithm is a function which assigns to any pair of states of the system (i.e., the initial state and the desired state) a continuous motion of the system starting at the initial state and ending at the desired state. A recent survey of algorithmic motion planning can be found in Sharir (1997); see also Latombe (1991).

Farber (2003; 2004) reveal the topological nature of the robot motion planning problem. They show that the navigational complexity of configuration spaces, $\mathbf{TC}(X)$, is a homotopy invariant quantity which can be studied using the algebraic

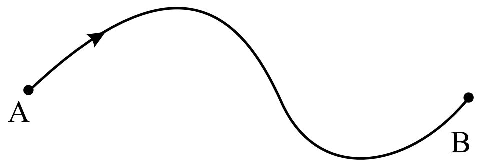

Figure 10.

topology tools. This theory explains how knowing the cohomology algebra of configuration space of a robot one may predict instabilities of its behavior.

Below in this article we describe the results of Farber (2003; 2004) adding some more recent developments.

Let X denote the configuration space of the mechanical system. Continuous motions of the system are represented by continuous paths $\gamma\colon [0,1] \to X$. Here the point $A = \gamma(0)$ represents the initial state and $\gamma(1) = B$ represents the final state of the system (Figure 10).

Assume that X is path-connected. Practically this means that one may fully control the system and bring it to an arbitrary state from any given state. Denote by PX the space of all continuous paths $\gamma\colon [0,1] \to X$. The space PX is supplied with the compact-open topology. Let

$$\pi\colon PX \to X \times X$$

be the map which assigns to a path γ the pair $(\gamma(0), \gamma(1)) \in X \times X$ of the initial-final configurations. π is a fibration in the sense of Serre.

DEFINITION 6.1. A motion planning algorithm is a section

$$s\colon X \times X \to PX \tag{3}$$

of fibration π, i.e.,

$$\pi \circ s = 1_{X \times X}. \tag{4}$$

The first question to ask is whether there exist motion planning algorithms which are continuous? Continuity of a motion planning algorithm s means that the suggested route $s(A,B)$ of going from A to B depends continuously on the states A and B.

LEMMA 6.2. *A continuous motion planning algorithm in X exists if and only if the space X is contractible.*[1]

Proof. Let $s\colon X \times X \to PX$ be a continuous MP algorithm. Here for $A, B \in X$ the image $s(A,B)$ is a path starting at A and ending at B. Fix $B = B_0 \in X$. Define $F(x,t) = s(x, B_0)(t)$. Here $F\colon X \times [0,1] \to X$ is a continuous deformation

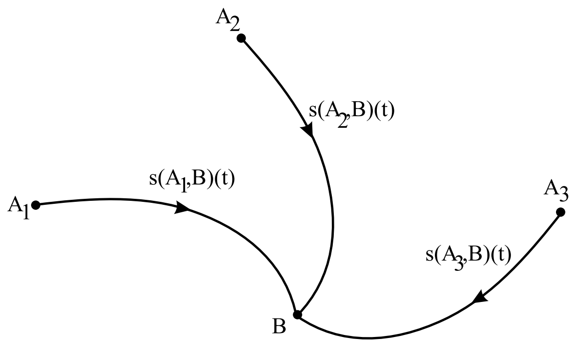

Figure 11.

with $F(x, 0) = x$ and $F(x, 1) = B_0$ for any $x \in X$. This shows that X must be contractible.

Conversely, let X be contractible. Then there exists a deformation $F: X \times [0, 1] \to X$ collapsing X to a point $B_0 \in X$. One may connect any two given points A and B by the concatenation of the path $F(A, t)$ and the inverse path to $F(B, t)$. $\square$

COROLLARY 6.3. *For a system with noncontractible configuration space any motion planning algorithm must be discontinuous.*

We see that the motion planning algorithms appearing in real situations are most likely discontinuous. Our main goal is to study the *discontinuities* in these algorithms. Having this goal in mind, with any path-connected topological space X we associate a numerical invariant $\mathbf{TC}(X)$ measuring the complexity of the problem of navigation in X. We give four different descriptions of how the number $\mathbf{TC}(X)$ influences the structure of the motion planning algorithms in X. One of these descriptions identifies $\mathbf{TC}(X)$ with the minimal number of "*continuous rules*" which are needed to describe a motion planning algorithm in X. On the other hand the number $\mathbf{TC}(X)$ equals the minimal "*order of instability*" of motion planning algorithms in X. We will also recover the number $\mathbf{TC}(X)$ while dealing with the *random motion planning algorithms* in X.

Formally we act in a different way: we define four a priori distinct notions of navigational complexity of topological spaces (which we denote $\mathbf{TC}_j(X)$, where $j = 1, 2, 3, 4$) and we show that $\mathbf{TC}_i(X) = \mathbf{TC}_j(X)$ for "good" spaces X (for example, for polyhedrons).

[1] This result was first observed in Farber (2003).

7. Tame motion planning algorithms

DEFINITION 7.1. A motion planning algorithm $s: X \times X \to PX$ is called tame if $X \times X$ can be split into finitely many sets

$$X \times X = F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_k \tag{5}$$

such that

1. $s|_{F_i}: F_i \to PX$ is continuous, $i = 1, \ldots, k$,
2. $F_i \cap F_j = \emptyset$, where $i \neq j$,
3. Each F_i is an Euclidean Neighborhood Retract (ENR).[2]

For a fixed pair of points $(A, B) \in F_i$, the curve produced by the algorithm $t \mapsto s(A, B)(t) \in X$ is a continuous curve in X which starts at point $A \in X$ and ends at point $B \in X$. This curve depends continuously on (A, B) assuming that the pair of points (A, B) varies in the set F_i.

Recall the definition of ENR:

DEFINITION 7.2. A topological space X is called an ENR if it can be embedded into an Euclidean space $X \subset \mathbb{R}^k$ such that for some open neighborhood $X \subset U \subset \mathbb{R}^k$ there exists a retraction $r: U \to X$, $r|_X = 1_X$.

All motion planning algorithms which appear in practice are tame. The configuration space X is usually a semi-algebraic set and the sets $F_j \subset X \times X$ are given by equations and inequalities involving real algebraic functions; thus they are semi-algebraic as well. In practical situations the functions $s|_{F_j}: F_j \to PX$ are real algebraic and hence they are continuous.

DEFINITION 7.3. The topological complexity of a tame motion planning algorithm (3) is the minimal number of domains of continuity k in a representation of type (5).

DEFINITION 7.4. The topological complexity $\mathbf{TC}_1(X)$ of a path-connected topological space X is the minimal topological complexity of motion planning algorithms in X.

Observation. $\mathbf{TC}_1(X) = 1$ if and only if X is an ENR and it is contractible.

We set $\mathbf{TC}_1(X) = \infty$ if X admits no tame motion planning algorithms.

EXAMPLE 7.5. Let us show that $\mathbf{TC}_1(S^n) = 2$ *for n odd and* $\mathbf{TC}_1(S^n) \leq 3$ *for n even.*

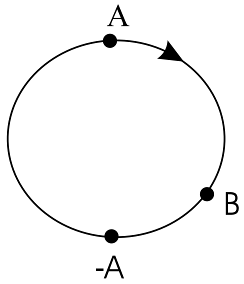

Figure 12.

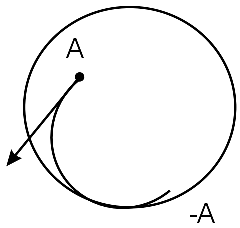

Figure 13.

Let $F_1 \subset S^n \times S^n$ be the set of all pairs (A, B) such that $A \neq -B$. We may construct a continuous section $s_1 : F_1 \to PS^n$ by moving A toward B along the shortest geodesic arc.

Consider now the set $F_2 \subset S^n \times S^n$ consisting of all pairs antipodal $(A, -A)$. If n is odd we may construct a continuous section $s_2 : F_2 \to PS^n$ as follows. Fix a nonvanishing tangent vector field v on S^n. Move A toward the antipodal point $-A$ along the semi-circle tangent to vector $v(A)$.

In the case when n is even find a tangent vector field v with a single zero $A_0 \in S^n$. Define $F_2 = \{(A, -A); A \neq A_0\}$ and define $s_2 : F_2 \to PS^n$ as above. The set $F_3 = \{(A_0, -A_0)\}$ consists of a single pair; define $s_3 : F_3 \to PS^n$ by choosing an arbitrary path from A_0 to $-A_0$.

8. The Schwarz genus

Let $p : E \to B$ be a fibration. Its Schwartz genus is defined as the minimal number k such that there exists an open cover of the base $B = U_1 \cup U_2 \cup \cdots \cup U_k$ with the property that over each set $U_j \subset B$ there exists a continuous section $s_j : U_j \to E$ of $E \to B$. This notion was introduced by A. S. Schwarz in 1958. In 1987–1988 S. Smale and V. A. Vassiliev applied the notion of Schwarz genus to study complexity of algorithms of solving polynomial equations.

[2] An equivalent concept was introduced in Farber (2004) under the name “motion planner.”

The genus of a fibration equals 1 if and only if it admits a continuous section.

The genus of the Serre fibration $P_0X \to X$ coincides with the Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of X, see Cornea et al. (2003). Here $P_0(X)$ is the space of paths in X which start at the base point $x_0 \in X$. For the motion planning problem we need to study a different fibration $\pi\colon PX \to X \times X$.

9. The second notion of topological complexity

The invariant $\mathbf{TC}_1(X)$ introduced above seems to be quite natural from the robotics point of view. However a more convenient topological invariant uses open covers instead of decompositions into ENR's.

DEFINITION 9.1. Let X be a path-connected topological space. The number $\mathbf{TC}_2(X)$ is defined as the Schwartz genus of the fibration

$$\pi\colon PX \to X \times X.$$

This notion coincides with the original definition of the topological complexity of the robot motion planning problem given in Farber (2003).

Explicitly, $\mathbf{TC}_2(X)$ is the minimal number k such that there exists an open cover

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

with the property that π admits a continuous section $s_j\colon U_j \to PX$ over each $U_j \subset X \times X$.

Note that the inclusion $U_j \to X \times X$ may be not null-homotopic. For example, if X is a polyhedron, there always exists a continuous section over a small neighborhood of the diagonal $X \subset X \times X$.

We know that $\mathbf{TC}_2(X) = 1$ if and only if X is contractible.

LEMMA 9.2. *One has*

$$\mathrm{cat}(X) \le \mathbf{TC}_2(X) \le \mathrm{cat}(X \times X).$$

Proof. We shall use two following simple properties of the Schwartz genus. Consider a fibration $E \to B$.

1. Let $B' \subset B$ be a subset, $E' = p^{-1}(B')$. Then the genus of $E' \to B'$ is less than or equal to the genus of $E \to B$.
2. The genus of $E \to B$ is less than or equal to $\mathrm{cat}(B)$.

To probe the lemma, apply 1 to the fibration $PX \to X \times X$ and to the subset $X \times x_0 \subset X \times X$. Note that $\pi^{-1}(X \times x_0) = P_0X$. We find $\mathbf{TC}_2(X) \ge \mathrm{cat}(X)$.

2 gives $\mathbf{TC}_2(X) \le \mathrm{cat}(X \times X)$. □

Exercise. Let G be a connected Lie group. Then

$$\mathbf{TC}_2(G) = \mathrm{cat}(G).$$

EXAMPLE 9.3. $\mathbf{TC}_2(\mathrm{SO}(3)) = \mathrm{cat}(\mathrm{SO}(3)) = \mathrm{cat}(\mathbb{R}\mathbf{P}^3) = 4$.

This example is important for robotics since SO(3) is the configuration space of a rigid body in $\mathbb{R}^3$ fixed at a point.

10. Homotopy invariance

THEOREM 10.1. *The number* $\mathbf{TC}_2(X)$ *is a homotopy invariant of* X.

See Farber (2003) for a proof.

11. Order of instability of a motion planning algorithm

Let $s: X \times X \to PX$ be a tame motion planning algorithm and let

$$X \times X = F_1 \cup F_2 \cup \ldots \cup F_k \tag{6}$$

be a decomposition into domains of continuity as in Definition 7.1. Here $F_i \cap F_j = \varnothing$ and each F_j is an ENR.

DEFINITION 11.1. The *order of instability* of the decomposition (6) is the maximal r so that for some sequence of indices

$$1 \leq i_1 < i_2 < \cdots < i_r \leq k$$

the intersection

$$\overline{F}_{i_1} \cap \overline{F}_{i_2} \cap \ldots \cap \overline{F}_{i_r} \neq \varnothing$$

in not empty. The *order of instability* of a motion planning algorithm[3] s is the minimal order of instability of decompositions (6) for s.

The order of instability is an important functional characteristic of a motion planning algorithm. If the order of instability equals r then for any $\epsilon > 0$ there exist r pairs of initial-final configurations

$$(A_1, B_1), \quad (A_2, B_2), \quad \ldots, \quad (A_r, B_r)$$

which are within distance $< \epsilon$ from one another and which lie in distinct sets F_i.

This means that small perturbations of the input data (A, B) may lead to r essentially distinct motions suggested by the motion planning algorithm.

DEFINITION 11.2. Let $\mathbf{TC}_3(X)$ be defined as the minimal order of instability of all tame motion planning algorithms in X.

[3] This notion was introduced and studied in Farber (2004).

Obviously one has:

$$\mathbf{TC}_3(X) \leq \mathbf{TC}_1(X). \tag{7}$$

12. Random motion planning algorithms

In this section we analyze complexity of random motion planning algorithms, following Farber (2005).

Let X be a path-connected topological space.

A *random n-valued path* σ in X starting at $A \in X$ and ending at $B \in X$ is given by an ordered sequence of paths $\gamma_1, \ldots, \gamma_n \in PX$ and an ordered sequence of real numbers $p_1, \ldots, p_n \in [0,1]$ such that each $\gamma_j: [0,1] \to X$ is a continuous path in X starting at $A = \gamma_j(0)$ and ending at $B = \gamma_j(1)$, and

$$p_j \geq 0, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{8}$$

One thinks of the paths $\gamma_1, \ldots, \gamma_n$ as being different *states* of σ (Figure 15).

The number p_j is the *probability* that the random path σ is in state γ_j. Random path σ as above will be written as a formal linear combination

$$\sigma = p_1\gamma_1 + p_2\gamma_2 + \cdots + p_n\gamma_n.$$

Equality between n-valued random paths is understood as follows: the random path

$$\sigma = p_1\gamma_1 + p_2\gamma_2 + \cdots + p_n\gamma_n.$$

is equal to

$$\sigma' = p'_1\gamma'_1 + p'_2\gamma'_2 + \cdots + p'_n\gamma'_n$$

iff $p_j = p'_j$ for all $j = 1, \ldots, n$ and, besides, $\gamma_j = \gamma'_j$ for all indices j with $p_j \neq 0$.

In other words, the path γ_j which appears with the zero probability $p_j = 0$ could be replaced by any other path starting at A and ending at B.

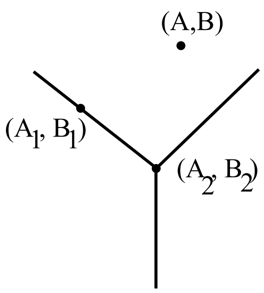

Figure 14.

We denote by P_nX the set of all n-valued random paths in X. The set P_nX has a natural topology: It is a factor-space of a subspace of the Cartesian product of n copies of $PX \times [0, 1]$.

The canonical map

$$\pi\colon P_nX \to X \times X \tag{9}$$

assigning to a random path its initial and end points is continuous.

DEFINITION 12.1. An *n-valued random motion planning algorithm* is defined as a continuous section

$$s\colon X \times X \to P_nX. \tag{10}$$

of fibration (9).

Given a pair $(A, B) \in X \times X$ (an input), the output of the random motion planning algorithm (10) is an ordered probability distribution

$$s(A, B) = p_1\gamma_1 + \cdots + p_n\gamma_n \tag{11}$$

supported on n paths between A and B. In other words, the algorithm s produces the motion γ_j with probability p_j where $j = 1, \ldots, n$.

Now we come to yet another notion of complexity of path-connected topological spaces:

DEFINITION 12.2. Let $\mathbf{TC}_4(X)$ be defined as the minimal integer n such that there exists an n-valued random motion planning algorithm $s\colon X \times X \to P_nX$.

13. Equality theorem

THEOREM 13.1. *Let X be a simplicial polyhedron. Then four notions of topological complexity introduced above coincide, i.e., one has*

$$\mathbf{TC}_1(X) = \mathbf{TC}_2(X) = \mathbf{TC}_3(X) = \mathbf{TC}_4(X). \tag{12}$$

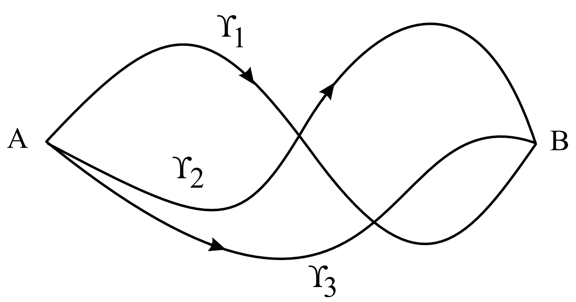

Figure 15.

Proof. Let k denote $\mathbf{TC}_1(X)$ and let $s: X\times X \to PX$ be a tame motion planning algorithm (as in Definition 7.1). We have a splitting $X \times X = F_1 \cup \cdots \cup F_k$ into disjoint ENRs such that the restriction $s|_{F_j}$ is continuous. Let us show that one may enlarge each set F_j to an open set $U_j \supset F_j$ such that the section $s|_{F_j}$ extends to a continuous section s'_j defined on U_j. This would prove that

$$\mathbf{TC}_2(X) \leq \mathbf{TC}_1(X). \tag{13}$$

We will use the following property of the ENRs: *If $F \subset X$ and both spaces F and X are ENRs then there is an open neighborhood $U \subset X$ of F and a retraction $r: U \to F$ such that the inclusion $j: U \to X$ is homotopic to $i \circ r$, where $i: F \to X$ denotes the inclusion*. See Dold, (1972, Chapter 4, Section 8) for a proof.

Using the fact that the sets F_i and $X \times X$ are ENRs, we find that there exists an open neighborhood $U_i \subset X \times X$ of the set F_i and a continuous homotopy $h^i_\tau: U_i \to X\times X$, where $\tau \in [0,1]$, such that $h^i_0: U_i \to X\times X$ is the inclusion and h^i_1 is a retraction of U_i onto F_i. We will describe now a continuous map $s'_i: U_i \to PX$ with $E \circ s'_i = 1_{U_i}$. Given a pair $(A, B) \in U_i$, the path $h^i_\tau(A, B)$ in $X \times X$ is a pair of paths (γ, δ), where γ is a path in X starting at the point $\gamma(0) = A$ and ending at a point $\gamma(1)$, and δ is a path in X starting at $B = \delta(0)$ and ending at $\delta(1)$. Note that the pair $(\gamma(1), \delta(1))$ belongs to F_i; therefore the motion planner $s_i: F_i \to PX$ defines a path

$$\xi = s_i(\gamma(1), \delta(1)) \in PX$$

connecting the points $\gamma(1)$ and $\delta(1)$. Now we set $s'_i(A, B)$ to be the concatenation of γ, ξ, and δ^{-1} (the reverse path of δ):

$$s'_i(A, B) = \gamma \cdot \xi \cdot \delta^{-1}.$$

Now we want to show that X always admits a tame motion planning algorithm (see Definition 7.1) with the number of local domains F_j equal to $\ell = \mathbf{TC}_2(X)$. This will show that

$$\mathbf{TC}_1(X) \leq \mathbf{TC}_2(X). \tag{14}$$

Let

$$U_1 \cup U_2 \cup \cdots \cup U_\ell = X \times X, \quad \text{where } \ell = \mathbf{TC}_2(X), \tag{15}$$

be an open cover such that for any $i = 1, \ldots, \ell$ there exists a continuous motion planning map $s_i: U_i \to PX$ with $\pi \circ s_i = 1_{U_i}$. Find a piecewise linear partition of unity $\{f_1, \ldots, f_\ell\}$ subordinate to the cover (15). Here $f_i: X \times X \to [0,1]$ is a piecewise linear function with support in U_i and such that for any pair $(A, B) \in X \times X$, it holds that

$$f_1(A, B) + f_2(A, B) + \cdots + f_\ell(A, B) = 1.$$

Fix numbers $0 < c_i < 1$ where $i = 1, \dots, \ell$ with $c_1 + \cdots + c_\ell = 1$. Let a subset $V_i \subset X \times X$, where $i = 1, \dots, \ell$, be defined by the following system of inequalities

$$\begin{cases} f_j(A,B) < c_j & \text{for all } j < i, \\ f_i(A,B) \geq c_i. \end{cases}$$

Then:

(a) each V_i is an ENR;

(b) V_i is contained in U_i; therefore, the section $s_i\colon U_i \to PX$ restricts onto V_i and defines a continuous section over V_i;

(c) the sets V_i are pairwise disjoint, $V_i \cap V_j = \varnothing$ for $i \neq j$;

(d) $V_1 \cup V_2 \cup \cdots \cup V_k = X \times X$.

Hence we see that the sets V_i and the sections $s_i|_{V_i}$ define a tame motion planning algorithm in the sense of Definition 7.1 with $\ell = \mathbf{TC}_2(X)$ local domains.

Now we prove that

$$\mathbf{TC}_3(X) \leq \mathbf{TC}_2(X). \tag{16}$$

Suppose that $s\colon X \times X \to PX$ is a tame motion planning algorithm with domains of continuity $F_1, \dots, F_k \subset X \times X$. Denote the order of instability of the decomposition $X \times X = F_1 \cup \cdots \cup F_k$ by $r \leq k$. Then any intersection of the form

$$\overline{F}_{i_1} \cap \cdots \cap \overline{F}_{i_{r+1}} = \varnothing, \tag{17}$$

is empty, where $1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq k$. For any index $i = 1, \dots, k$ fix a continuous function $f_i\colon X \times X \to [0,1]$ such that $f_i(A,B) = 1$ if and only if the pair (A,B) belongs to $\overline{F}_i$ and such that the support $\operatorname{supp}(f_i)$ retracts onto F_i. Let $\phi\colon X \times X \to \mathbb{R}$ be the maximum of (finitely many) functions of the form $f_{i_1} + f_{i_2} + \cdots + f_{i_{r+1}}$ for all increasing sequences $1 \leq i_1 < i_2 < \cdots < i_{r+1} \leq k$ of length $r + 1$. We have:

$$\phi(A,B) < r + 1$$

for any pair $(A,B) \in X \times X$, as follows from (17).

Let $U_i \subset X \times X$ denote the set of all (A,B) such that

$$(r+1) \cdot f_i(A,B) > \phi(A,B).$$

Then U_i is open and contains $\overline{F}_i$, and hence the sets $U_1, \dots, U_k$ form an open cover of $X \times X$. On the other hand, any intersection

$$U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_{r+1}} = \varnothing$$

is empty.

As above we may assume that the sets $U_1, \dots, U_k$ are small enough so that over each U_i there exists a continuous motion planning section (here we use the assumption that each F_i is an ENR). Applying Lemma 13.2 (see below) we conclude that $\mathbf{TC}_2(X) \le r$.

Combining inequalities (7), (13), (14), (16) we obtain $\mathbf{TC}_1(X) = \mathbf{TC}_2(X) = \mathbf{TC}_3(X)$.

Next we show that $\mathbf{TC}_2(X) = \mathbf{TC}_4(X)$. This last argument is an adjustment of the proof of Schwarz (1966, Proposition 2).

Assume that there exists an n-valued random motion planning algorithm $s: X\times X \to P_nX$ in X. The right-hand side of formula (11) defines continuous real valued functions $p_j: X \times X \to [0,1]$, where $j = 1, \dots, n$. Let U_j denote the open set $p_j^{-1}(0,1] \subset X \times X$. The sets $U_1, \dots, U_n$ form an open covering of $X \times X$. Setting $s_j(A,B) = \gamma_j$, one gets a continuous map $s_j : U_j \to PX$ with $\pi \circ s_i = 1_{U_j}$. Hence, $n \ge \mathbf{TC}_2(X)$ according to the definition of $\mathbf{TC}_2(X)$.

Conversely, setting $k = \mathbf{TC}_2(X)$, we obtain that there exists an open cover $U_1, \dots, U_k \subset X\times X$ and a sequence of continuous maps $s_i: U_i \to PX$ where $\pi\circ s_i = 1_{U_i}$, $i = 1, \dots, k$. Extend s_i to an arbitrary (possibly discontinuous) mapping

$$S_i: X \times X \to PX$$

satisfying $\pi \circ S_i = 1_{X\times X}$. This can be done without any difficulty; it amounts in making a choice of a connecting path for any pair of points $(A,B) \in X\times X - U_i$. One may find a continuous partition of unity subordinate to the open cover $U_1, \dots, U_k$. It is a sequence of continuous functions $p_1, \dots, p_k: X \times X \to [0,1]$ such that for any pair $(A,B) \in X \times X$ one has

$$p_1(A,B) + p_2(A,B) + \cdots + p_k(A,B) = 1$$

and the closure of the set $p_i^{-1}(0,1]$ is contained in U_i. We obtain a continuous k-valued random motion planning algorithm $s: X\times X \to P_nX$ given by the following explicit formula

$$s(A,B) = p_1(A,B)S_1(A,B) + \cdots + p_k(A,B)S_k(A,B). \tag{18}$$

The continuity of s follows from the continuity of the maps S_i restricted to the domains $p_i^{-1}(0,1]$. This completes the proof. □

LEMMA 13.2. *Let X be a path-connected metric space. Consider an open cover $X\times X = U_1 \cup U_2 \cup \cdots \cup U_\ell$ such that for any $i = 1, \dots, \ell$ there exists a continuous map $s_i: U_i \to PX$ with $\pi\circ s_i = 1_{U_i}$. Suppose that for some integer r any intersection*

$$U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_r} = \varnothing$$

is empty where $1 \le i_1 < i_2 < \cdots < i_r \le \ell$. Then $\mathbf{TC}_2(X) < r$.

A proof of Lemma 13.2 can be found in Farber (2004).

Although the numbers $\mathbf{TC}_j(X)$ (where $j = 1, 2, 3, 4$) coincide when X is a simplicial polyhedron, they do not coincide when X is a general topological space. The most convenient notion topologically is $\mathbf{TC}_2(X)$.

Notation. In what follows we will use the notation $\mathbf{TC}(X) = \mathbf{TC}_2(X)$.

14. An upper bound for TC(*X*)

THEOREM 14.1. *For any path-connected paracompact locally contractible space X one has*

$$\mathbf{TC}(X) \leq 2 \dim X + 1, \tag{19}$$

where dim X *denotes the covering dimension of* X.

Proof. We know that $\mathbf{TC}(X) \leq \operatorname{cat}(X \times X)$. Combine this with $\operatorname{cat}(X \times X) \leq \dim(X \times X) + 1 = 2 \dim X + 1$. □

This result can be improved assuming that X is highly connected:

THEOREM 14.2. *If X is an r-connected CW-complex then*

$$\mathbf{TC}(X) < \frac{2 \cdot \dim X + 1}{r + 1} + 1. \tag{20}$$

See Farber (2004) for a proof.

15. A cohomological lower bound for TC(*X*)

In this section we describe a result from Farber (2003).

Let $\Bbbk$ be a field. The cohomology $H^*(X; \Bbbk) = H^*(X)$ is a graded $\Bbbk$-algebra with the multiplication

$$\cup : H^*(X) \otimes H^*(X) \to H^*(X)$$

given by the cup-product. The tensor product $H^*(X) \otimes H^*(X)$ is again a graded $\Bbbk$-algebra with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1| \cdot |u_2|} u_1 u_2 \otimes v_1 v_2.$$

Here $|v_1|$ and $|u_2|$ denote the degrees of cohomology classes v_1 and u_2 correspondingly. The cup-product $\cup$ is an algebra homomorphism.

DEFINITION 15.1. The kernel of the homomorphism

$$\cup : H^*(X) \otimes H^*(X) \to H^*(X)$$

is called the ideal of the zero-divisors of $H^*(X)$. The zero-divisors-cup-length of $H^*(X)$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X)$.

THEOREM 15.2. $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of* $H^*(X)$.

See Farber (2003) for a proof.

16. Examples

EXAMPLE 16.1. Consider the case $X = S^n$. Let $u \in H^n(S^n)$ be the fundamental class, and let $1 \in H^0(S^n)$ be the unit. Then

$$a = 1 \otimes u - u \otimes 1 \in H^*(S^n) \otimes H^*(S^n)$$

is a zero-divisor. Another zero-divisor is $b = u \otimes u$. Computing $a^2 = a \cdot a$ we find

$$a^2 = ((-1)^{n-1} - 1) \cdot u \otimes u.$$

Hence $a^2 = -2b$ for n even and $a^2 = 0$ for n odd.

We conclude: *the zero-divisors-cup-length of* $H^*(S^n; \mathbb{Q})$ *equals* 1 *for* n *odd and* 2 *for* n *even*. Applying Theorem 15.2 we find that $\mathbf{TC}(S^n) \geq 2$ for n odd and $\mathbf{TC}(S^n) \geq 3$ for n even. In Section 7 we constructed explicit motion planning algorithms having topological complexity 2 for n odd and 3 for n even. Hence,

$$\mathbf{TC}(S^n) = \begin{cases} 2, & \text{if } n \text{ is odd,} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

EXAMPLE 16.2. Here we calculate the number $\mathbf{TC}(X)$ when X is a graph.

THEOREM 16.3. *If X is a connected finite graph then*

$$\mathbf{TC}(X) = \begin{cases} 1, & \text{if } b_1(X) = 0, \\ 2, & \text{if } b_1(X) = 1, \\ 3, & \text{if } b_1(X) > 1. \end{cases}$$

Proof. If $b_1(X) = 0$ then X is contractible and hence $\mathbf{TC}(X) = 1$. If $b_1(X) = 1$ then X is homotopy equivalent to the circle and therefore $\mathbf{TC}(X) = \mathbf{TC}(S^1) = 2$, see above. Assume now that $b_1(X) > 1$. Then there exist two linearly independent classes $u_1, u_2 \in H^1(X)$. Thus

$$1 \otimes u_i - u_i \otimes 1, \quad i = 1, 2$$

are zero-divisors and their product equals $u_2 \otimes u_1 - u_1 \otimes u_2 \neq 0$ which implies $\mathbf{TC}(X) \geq 3$. On the other hand, we know that $\mathbf{TC}(X) \leq 3$ by Theorem 14.1. This completes the proof. □

EXAMPLE 16.4. Let $X = \Sigma_g$ be a compact orientable surface of genus g. Then

$$\mathbf{TC}(X) = \begin{cases} 3, & \text{if } g \leq 1, \\ 5, & \text{if } g > 1. \end{cases}$$

We leave the proof as an exercise.

17. Simultaneous control of many systems

Suppose that we have to control two systems simultaneously. We assume that the systems do not interact, i.e., the admissible states of one of the systems do not depend on the state of the other. Let X and Y be the corresponding configuration spaces. If we view these two systems as a new single system then the configuration space is the product $X \times Y$. For the topological complexity of the product one has the inequality:

THEOREM 17.1. $\mathbf{TC}(X \times Y) \leq \mathbf{TC}(X) + \mathbf{TC}(Y) - 1$.

A proof can be found in Farber (2003).

Suppose now that one has to control simultaneously n systems having configuration spaces $X_1, \dots, X_n$. The total configuration space is the Cartesian product

$$Y_n = X_1 \times X_2 \times \dots \times X_n. \tag{21}$$

We ask: *What is the asymptotics of the topological complexity* $\mathbf{TC}(Y_n)$ *for large* n?

We shall assume that the topological complexity of the space X_n is bounded, i.e., there exists a constant $M \geq 1$ such that $\mathbf{TC}(X_n) \leq M$ for all n. Applying Theorem 17.1 one obtains the inequality

$$\mathbf{TC}(Y_n) \leq n \cdot [M - 1] + 1. \tag{22}$$

This shows that the sequence $\mathbf{TC}(Y_n)$ growths at most linearly.

Let us assume additionally that each space X_n is path-connected and *homologically nontrivial*, i.e., $H^*(X_n) \neq H^*(\text{pt})$. Then one has

$$\mathbf{TC}(Y_n) \geq n + 1. \tag{23}$$

Proof. Let $u_r \in H^{i_r}(X_r)$ be a nontrivial class, where $i_r > 0$. Denote

$$w_r = 1 \times 1 \times \cdots \times u_r \times 1 \times \cdots \times 1 \in H^{i_r}(Y_n)$$

(here u_r stands on the r-th place). Then

$$\prod_{j=1}^{n} w_j \in H^*(Y_n)$$

is a nonzero class. The class

$$\overline{w}_j = w_j \otimes 1 - 1 \otimes w_j, \quad j = 1, \dots, n$$

is a zero-divisor and the product

$$\prod_{j=1}^{n} \overline{w}_j = \left(\prod_{j=1}^{n} w_j \right) \otimes 1 \pm \dots \neq 0$$

is nonzero. This proves (22) as follows from the cohomological lower bound. □

Combining the inequalities (22) and (23) one obtains:

COROLLARY 17.2. *Assume that each space X_r is path-connected and homologically nontrivial and the topological complexity* $\mathbf{TC}(X_r)$ *is bounded above. Then the topological complexity of the product Y_n* (*see* (21)) (*viewed as a function of n*) *has a linear growth. In particular, for any finite-dimensional path-connected and homologically nontrivial polyhedron X the sequence* $\mathbf{TC}(X^n)$ *as a function of n has a linear growth.*

This result has an important implication in the control theory:

THEOREM 17.3. *A centralized control by n identical independent systems has topological complexity which is linear in n* (*more precisely, the inequalities* (22) *and* (23) *are satisfied*). *The distributed control, i.e., when each of the objects is controlled independently of the others, has an exponential topological complexity* $\mathbf{TC}(X)^n$.

We see that in practical situations the centralized control by many independent objects could be organized so that its “much more stable” than the distributed control.

18. Another inequality relating TC(X) to the usual category

The result of this section was inspired by a discussion with H.-J. Baues.

Consider the fibration $\pi\colon PX \to X \times X$, cf. Definition 6.1.

LEMMA 18.1. *Let $U \subset X \times X$ be a subset. There exists a continuous section $s\colon U \to PX$, $\pi \circ s = 1_U$ of π over U if and only if the inclusion $U \to X \times X$ is homotopic to a map with values in the diagonal $\Delta X \subset X \times X$.*

Proof. Let $s\colon U \to PX$ be a section. Here $s(A,B)(t) \in X$ is a continuous function of A, B, t (where $(A,B) \in U$ and $t \in [0,1]$) such that $s(A,B)(0) = A$ and $s(A,B)(1) = B$. Define

$$\sigma\colon U \times [0,1] \to X \times X$$

by $\sigma(A,B)(t) = (s(A,B)(t), B)$. Then one has $\sigma(A,B)(0) = (A,B)$ and $\sigma(A,B)(1) = (B,B)$ takes values on the diagonal ΔX. Hence σ is a homotopy between the inclusion $U \to X \times X$ and a map with values on the diagonal.

Conversely, suppose that $\sigma_t\colon U \to X \times X$ is a homotopy from the inclusion to a map with values on the diagonal. Then $t \mapsto \sigma_t(A,B)$ is a path in $X \times X$ which starts at (A,B) and ends at a point (C,C). In other words, $\sigma_t(A,B)$ is a pair (γ_1, γ_2)

of paths in X where γ_1 starts at A, γ_2 starts at B, and the end points of these paths coincide. Hence the path $s = \gamma_1\gamma_2^{-1} \in PX$ is well-defined, continuously depends on A and B and starts at A and ends at B. We obtain a continuous section of π over U. □

COROLLARY 18.2. *The topological complexity* $\mathbf{TC}(X)$ *is the smallest* k *such that* $X \times X$ *can be covered by* k *open subsets* $U_1 \cup U_2 \cup \cdots \cup U_k = X \times X$ *such that each* $U_j \to X \times X$ *is homotopic to a map with values in the diagonal* $\Delta X \subset X \times X$.

The following inequality complements Lemma 9.2.

LEMMA 18.3. *If* X *is an ENR then*

$$\mathbf{TC}(X) \geq \mathrm{cat}((X \times X)/\Delta X) - 1.$$

Proof. Let $X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$ and each $U_i \to X \times X$ is homotopic to a map with values in $\Delta(X)$. Let $U'_j = U_j - \Delta(X)$ and $U''_j \subset (X \times X)/\Delta(X)$ be the image of U'_j under the canonical map $X \times X \to X \times X/\Delta X$. Then U''_j is null-homotopic and these sets cover the whole $X \times X/\Delta X$ except the base point of the factor-space. Hence, adding a contractible neighborhood of the base point gives a categorical cover of the factor-space. Existence of such neighborhood follows from the ENR assumption. This completes the proof. □

19. Topological complexity of bouquets

It is quite obvious that

$$\mathbf{TC}(X \vee Y) \geq \max\{\mathbf{TC}(X), \mathbf{TC}(Y)\}. \tag{24}$$

We shall prove the following:

THEOREM 19.1. *Let* X *and* Y *be two polyhedrons. Then* $\mathbf{TC}(X \vee Y)$ *is less than or equal to*

$$\max\{\mathbf{TC}(X), \mathbf{TC}(Y), \mathrm{cat}(X) + \mathrm{cat}(Y) - 1\}. \tag{25}$$

Proof. The product $(X \vee Y) \times (X \vee Y)$ is a union of four spaces

$$X \times X, \quad Y \times Y, \quad X \times Y, \quad Y \times X$$

and any two of these spaces intersect at a single point (p, p) where p is the join point of the wedge $X \vee Y$. Over each of these sets one may construct a motion planning algorithm having respectively

$$\mathbf{TC}(X), \quad \mathbf{TC}(Y), \quad \mathrm{cat}(X) + \mathrm{cat}(Y) - 1, \quad \mathrm{cat}(X) + \mathrm{cat}(Y) - 1$$

domains of continuity. For example, the algorithm over $X \times Y$ takes pairs $(x, y) \in X \times Y$ as an input and finds a path α in X connecting x with p, a path β in Y connecting y with p and finally produces the path $\alpha\beta^{-1}$ as the output. To make the choice of α continuous one has to split X into $\mathrm{cat}(X)$ pieces; to make the choice of β continuous one splits Y into $\mathrm{cat}(Y)$ pieces. Similarly to the proof of the product inequality (see Farber, 2003, Theorem 11) one may rearrange the totality of $\mathrm{cat}(X) \times \mathrm{cat}(Y)$ products into $\mathrm{cat}(X) + \mathrm{cat}(Y) - 1$ sets (the checkerboard trick) such that the algorithm is continuous over each of them.

The remaining arguments of the proof are similar (compare with the next section), we leave them as an exercise for the reader. □

20. A general recipe to construct a motion planning algorithm

Let X be a path-connected polyhedron and let $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ be a *nice* open cover of X with the property that each inclusion $U_i \to X$ is null-homotopic. The word "*nice*" means that the Main assumption (see below) is satisfied. Our goal is *to construct a motion planning algorithm in X with* $2m - 1$ *local domains where m is the multiplicity of the covering* $\mathcal{U}$, i.e., the maximal number of distinct domains U_j having a nonempty intersection.

Introduce subsets $V_1, V_2, \ldots, V_m$ where $V_r \subset X$ denotes the set of points $x \in X$ which are covered by precisely r sets U_j.

Main assumption. Each V_i is an ENR.

For any multi-index $\alpha = (1 \le i_1 < i_2 < \cdots < i_r \le n)$ denote

$$U_\alpha = \bigcap_{k=1}^{r} U_{i_k}.$$

Then

$$V_r = \bigcup_{|\alpha|=r} U_\alpha - \bigcup_{|\alpha|=r+1} U_\alpha. \tag{26}$$

Note that $V_r = \emptyset$ for $r > m$.

LEMMA 20.1.

(A) *Each set* $W_\alpha = U_\alpha \cap V_r$ (*where* $|\alpha| = r$) *is closed and open in* V_r.

(B) *The sets* W_α *and* W_β *are disjoint for* $\alpha \neq \beta$, $|\alpha| = r = |\beta|$.

Proof. Clearly $x \in U_\alpha \cap U_\beta$ implies that $x \in U_{\alpha\cup\beta}$. This implies statement (B).

Now we want to show slightly more, namely, that the sets $\overline{W}_\alpha$ and W_β are disjoint for $|\alpha| = r = |\beta|$, $\alpha \neq \beta$. Indeed, if $x \in \overline{W}_\alpha \cap W_\beta$ then $x = \lim x_n$ where $x_n \in W_\alpha$. Since x lies in W_β one has $x_n \in W_\beta$ for all large n and hence $x_n \in U_{\alpha\cup\beta} \not\subset V_r$ — a contradiction.

These two statements imply that $\overline{W}_\alpha \cap V_r = W_\alpha$ i.e., W_α is closed in V_r. As follows from the definition W_α is also open in V_r. □

LEMMA 20.2. *One has*

$$\overline{V}_r \subset \bigcup_{k\leq r} V_k. \tag{27}$$

Proof. It follows directly from (26). □

LEMMA 20.3. *Over each set $V_r \times V_{r'} \subset X \times X$ one may construct an explicit continuous section of the fibration $\pi\colon PX \to X \times X$.*

Proof. We know that $V_r = \bigcup_{|\alpha|=k} W_\alpha$ and each W_α is open and closed in V_r. Hence it is enough to construct a continuous section over each $W_\alpha \times W_\beta$. Let i and j be the smallest indices appearing in the multi-indices α and β correspondingly. Then $W_\alpha \subset U_i$ and $W_\beta \subset U_j$. Let $H^i_t\colon U_i \times I \to X$ and $H^j_t\colon U_j \times I \to X$ be the homotopies contracting U_i and U_j to the base point $x_0 \in X$. Then, given a pair $(x, y) \in W_\alpha \times W_\beta$ one constructs a path connecting them as follows: it is concatenation of the path $H^i_t(x)$ leading from x to the base point and then follows the reverse path to $H^j_t(y)$. □

Denote

$$A_k = \bigcup_{r+r'=k+1} V_r \times V_{r'} \subset X \times X, \tag{28}$$

where $k = 1, 2, \ldots, 2m - 1$. These sets are ENR's (by the assumption) and cover $X \times X$.

LEMMA 20.4. *Each product $V_r \times V_{r'}$, where $r + r' = k + 1$, is closed and open in A_k.*

Proof. It follows from (20.2). □

Hence the described above local sections over each $V_r \times V_{r'}$ combine into a continuous section over A_k. In total, we have $2m - 1$ local sections.

21. How difficult is to avoid collisions in $\mathbb{R}^m$?

In this section we start discussing the problem of finding the topological complexity $\mathbf{TC}(F(\mathbb{R}^m, n))$ of the configuration space $F(\mathbb{R}^m, n)$ of n distinct points in the

Euclidean space $\mathbb{R}^m$. A motion planning algorithm in $F(\mathbb{R}^m, n)$ takes as an input two configurations of n distinct points in $\mathbb{R}^m$ and produces n continuous curves $A_1(t), \ldots, A_n(t) \in \mathbb{R}^m$, where $t \in [0,1]$, such that $A_i(t) \neq A_j(t)$ for all $t \in [0,1]$, $i \neq j$ and $(A_1(0), \ldots, A_n(0))$ and $(A_1(1), \ldots, A_n(1))$ are the first and the second given configurations. In other words, a motion planning algorithm in $F(\mathbb{R}^m, n)$ moves one of the given configurations into another avoiding collisions.

The following theorem was obtained in Farber and Yuzvinsky (2004).

THEOREM 21.1. *One has*

$$\mathbf{TC}(F(\mathbb{R}^m, n)) = \begin{cases} 2n-1 & \text{for any odd } m, \\ 2n-2 & \text{for } m = 2. \end{cases}$$

At the moment we do not know the answer for the case $m \geq 4$ even. We know that in this case the number $\mathbf{TC}(F(\mathbb{R}^m, n))$ is either $2n-1$ or $2n-2$.

Conjecture. For any m even one has $\mathbf{TC}(F(\mathbb{R}^m, n)) = 2n-2$.

We will give here some ideas of the proof of Theorem 21.1 referring the reader to Farber and Yuzvinsky (2004) for details. We will also discuss the possible approaches to construct explicit motion planning algorithms in $F(\mathbb{R}^m, n)$. Such algorithms could be useful in situations when a large number of objects must be moved automatically (without human intervention) from one position to another avoiding collisions.

Consider the set

$$H_{ij} = \{(y_1, \ldots, y_n); y_i \in \mathbb{R}^m, y_i = y_j\} \subset \mathbb{R}^{nm}.$$

Here $i, j \in \{1, 2, \ldots, n\}$, $i < j$. The set H_{ij} is a linear subspace of $\mathbb{R}^{nm}$ of codimension m. The system of subspaces $\{H_{ij}\}_{i<j}$ is an arrangement of linear subspaces of codimension m. Our approach to the problem is to view the union

$$H = \bigcup_{i<j} H_{ij}$$

as the set of obstacles:

$$F(\mathbb{R}^m, n) = \mathbb{R}^{nm} - H.$$

22. The case $m = 2$

Assume first that $m = 2$. This means that we are dealing with n distinct particles on the plane. Then $H_{ij} \subset \mathbb{C}^n$ is a complex subspace of codimension 1.

Consider a slightly more general situation. Let $\mathcal{A} = \{H\}$ be a finite set of hyperplanes in an affine complex space $\mathbb{C}^n$. Denote by $M(\mathcal{A})$ the complement

$$M(\mathcal{A}) = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H.$$

We will study the motion planning problem in $M(\mathcal{A})$. We may say that we live in $\mathbb{C}^n$ and the union of hyperplanes $\bigcup H$ represent our obstacles.

Recall some terminology from the theory of arrangements (Orlik and Terao, 1992). If $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$ then $\mathcal{A}$ is called *central*, and up to change of coordinates the hyperplanes can be assumed linear. Suppose that $\mathcal{A}$ is linear. For each $H \in \mathcal{A}$ one can fix a linear functional α_H (unique up to a non-zero multiplicative constant) such that $H = \{\alpha_H = 0\}$. A set of hyperplanes $H_i \in \mathcal{A}$ is called *linear independent* if the corresponding functionals α_{H_i} are linearly independent. The rank of $\{\alpha_H\}$, i.e., the cardinality of a maximal independent subset, is called the *rank* of $\mathcal{A}$ and denoted by $\mathrm{rk}(\mathcal{A})$. Clearly $\mathrm{rk}(\mathcal{A}) \leq n$ and the equality occurs if and only if $\bigcap_H H = 0$.

If $\mathcal{A}$ is not central we define its rank as the rank of a maximal central subarrangement of $\mathcal{A}$.

While dealing with the arrangement complements we will need the following nontrivial result (Orlik and Terao, 1992): *if $\mathcal{A}$ is an arbitrary arrangement of rank r then the complement $M(\mathcal{A})$ has homotopy type of a simplicial complex of dimension r.*

Note that the rank of the braid arrangement $\{H_{ij}\}_{i<j}$ in $\mathbb{C}^n$ equals $n - 1$.

COROLLARY 22.1. *The configuration space $F(\mathbb{C}, n)$ has homotopy type of a simplicial complex of dimension $n - 1$.*

Combining this with Theorems 10.1 and 14.1 we obtain:

COROLLARY 22.2. $\mathbf{TC}(F(\mathbb{C}, n)) \leq 2n - 1$.

This result can be improved:

THEOREM 22.3. *Let $\mathcal{A}$ be a central complex hyperplane arrangement of rank r. Then the topological complexity of the complement $M(\mathcal{A})$ is less or equal than $2r$. In particular one has $\mathbf{TC}(F(\mathbb{C}, n)) \leq 2n - 2$.*

Proof. Let $\mathcal{A}$ be $\{H_1, \ldots, H_\ell\} \subset \mathbb{C}^n$. Let H_1^* be a parallel copy of H_1 which is disjoint from H_1. Then the intersections

$$H_i \cap H_1^*, \quad i = 2, 3, \ldots, \ell$$

form a (in general, non-central) hyperplane arrangement $\mathcal{A}^*$ in $H_1^* \simeq \mathbb{C}^{n-1}$ of rank $r - 1$. There is a principal $\mathbb{C}^*$-fibration $M(\mathcal{A}) \to M(\mathcal{A}^*)$. The inclusion $M(\mathcal{A}^*) \subset$

$M(\mathcal{A})$ is a section of it. Hence the fibration is trivial $M(\mathcal{A}) \simeq M(\mathcal{A}^*) \times \mathbb{C}^*$ and using the product inequality (see Theorem 17.1) we find

$$\begin{aligned} \mathbf{TC}(M(\mathcal{A})) &\leq \mathbf{TC}(M(\mathcal{A}^*)) + \mathbf{TC}(\mathbb{C}^*) - 1 \\ &\leq [2(r-1)+1] + 2 - 1 = 2r. \end{aligned} \qquad \square$$

The opposite inequality requires an additional geometric property of the arrangement:

THEOREM 22.4. *Let $\mathcal{A}$ be a complex central hyperplane arrangement of rank r. Assume that there exist $2r-1$ hyperplanes $H_1, H_2, \dots, H_{2r-1}$ such that H_1, $H_2, \dots, H_r$ are independent and for any $1 \leq j \leq r$ the hyperplanes H_j, H_{r+1}, $H_{r+2}, \dots, H_{2r-1}$ are independent. Then one has $\mathbf{TC}(M(\mathcal{A})) \geq 2r$.*

The proof (see Farber and Yuzvinsky, 2004) uses the cohomological lower bound for the topological complexity and combinatorics of Orlik – Solomon algebras.

EXAMPLE 22.5. Consider the braid arrangement $\{H_{ij}\}_{i<j} \subset \mathbb{C}^n$. Here $r = n-1$ and $2r-1 = 2n-3$. We have $2n-3$ hyperplanes:

$$H_{12}, H_{13}, \dots, H_{1n}, H_{23}, H_{24}, \dots, H_{2n}$$

satisfying the condition of the above theorem.

COROLLARY 22.6. *One has $\mathbf{TC}(F(\mathbb{R}^2, n)) = 2n-2$.*

23. $\mathbf{TC}(F(\mathbb{R}^m, n))$ in the case $m \geq 3$ odd

Assume that $m \geq 3$ is odd. Then $F(\mathbb{R}^m, n)$ is $(m-2)$-connected and in particular it is simply connected. Its cohomology algebra is generated by the cohomology classes

$$e_{ij} \in H^{m-1}(F(\mathbb{R}^m, n)), \quad i \neq j$$

which arise as follows. Consider the map

$$\phi_{ij}\colon F(\mathbb{R}^m, n) \to S^{m-1}, \quad (y_1, y_2, \dots, y_n) \mapsto \frac{y_i - y_j}{|y_i - y_j|} \in S^{m-1}.$$

Then

$$e_{ij} = \phi_{ij}^*[S^{m-1}]$$

where $[S^{m-1}]$ is the fundamental class of the sphere S^{m-1}.

The cohomology classes e_{ij} satisfy the following relations:

$$e_{ij}^2 = 0, \quad \text{and} \quad e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = 0 \tag{29}$$

for any triple i, j, k. It follows that a product $e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}$ is nonzero if and only if the subgraph of the full graph on vertices $\{1, 2, \ldots, n\}$ having the edges (i_r, j_r) contains no cycles.

Hence for $m \geq 3$ the configuration space $F(\mathbb{R}^m, n)$ has homotopy type of a polyhedron of dimension $\leq (n-1)(m-1)$. Since it is $(m-2)$-connected we may use inequality (20) of Theorem 14.2 to find

$$\mathbf{TC}(F(\mathbb{R}^m, n)) < \frac{2(n-1)(m-1)+1}{m-1} + 1 = 2n - 1 + \frac{1}{m-1}.$$

We obtain:

COROLLARY 23.1. $\mathbf{TC}(F(\mathbb{R}^m, n)) \leq 2n - 1$.

We want to show that an equality holds in Corollary!23.1. We shall use the cohomological lower bound (see Theorem 15.2). Set $\overline{e_{ij}} = 1 \otimes e_{ij} - e_{ij} \otimes 1$. It is a zero-divisor of the cohomology algebra. Note that $(\overline{e_{ij}})^2 = -2 \cdot e_{ij} \otimes e_{ij} \neq 0$. Here we use the assumption that m is odd.

Consider the following product

$$\pi = \prod_{i=2}^{n} (\overline{e}_{1i})^2 \in A \otimes A.$$

We find $\pi = (-2)^{n-1} m \otimes m$, where

$$m = \prod_{i=2}^{n} e_{1i}.$$

The monomial $m \neq 0$ is nonzero and hence the product π is nonzero.

Using the cohomological lower bound for the topological complexity we obtain the opposite inequality $\mathbf{TC}(M) \geq 2n - 1$. This completes the proof of Theorem 21.1 in the case $m \geq 3$ odd.

24. Shade

Let $X \subset \mathbb{R}^n$ be a closed subset with connected complement $\mathbb{R}^n - X$. Our purpose is to find (or to estimate) the number $\mathbf{TC}(\mathbb{R}^n - X)$. Our main motivation is the special case when $X = \cup H$ is the union of finitely many affine subspaces.

DEFINITION 24.1. Let $v \in S^{n-1}$ be a unit vector. The *shade* of X in the direction of v is defined as

$$\mathrm{Shade}_v(X) = \{x + \lambda v; x \in X, \lambda \in \mathbb{R}_+\}. \tag{30}$$

In other words we assume that the light illuminating the space arrives from direction of vector v and that X is not transparent. Then $\mathrm{Shade}_v(X)$ is precisely the shaded parts of the space.

Assume that $X \subset \mathbb{R}^n$ satisfies the following condition: *For any point $p \in \mathbb{R}^n$ and for any generic unit vector $v \in S^{n-1}$ the distance*

$$\mathrm{dist}(p - \lambda v, X) \to +\infty \tag{31}$$

tends to $+\infty$ as λ tends to $+\infty$.

This condition is satisfied in two cases which are particularly important for us: when either X is compact or X is a union of finitely many affine subspaces. If X is a union of finitely many affine subspaces then the condition above is satisfied assuming that the vector v is not parallel to any of the subspaces.

LEMMA 24.2. *If* (31) *is satisfied then for a generic nonzero $v \in \mathbb{R}^n$ the distance*

$$\mathrm{dist}(p - \lambda v, \mathrm{Shade}_v(X))$$

tends to $+\infty$ as $\lambda \to +\infty$.

LEMMA 24.3. *If* (31) *is satisfied then for a generic nonzero vector v the complement of the shade $\mathbb{R}^n - \mathrm{Shade}_v(X)$ is contractible.*

Proof. We will show that any compact set $K \subset \mathbb{R}^n - \mathrm{Shade}_v(X)$ is null-homotopic in the complement $\mathbb{R}^n - \mathrm{Shade}_v(X)$. Assume that K is contained in a ball with center $p \in \mathbb{R}^n$ and radius $A > 0$. Using Lemma 24.2 find λ_A such that the distance between $\mathrm{Shade}_v(X)$ and $p - \lambda_A v$ is greater than A. The homotopy $h_t : K \to \mathbb{R}^n - \mathrm{Shade}_v(X)$, $t \in [0, 1]$, where $h_t(x) = x - \lambda t v$, takes K into the ball with center $p - \lambda_A v$ of radius A which is disjoint from $\mathrm{Shade}_v(X)$ and hence the image $h_1(K)$ can be contracted to a point in this ball. □

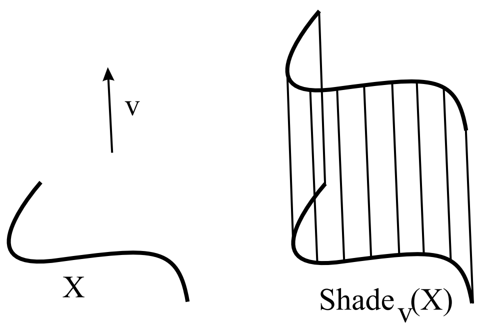

Figure 16.

DEFINITION 24.4. Let $X \subset \mathbb{R}^n$ be a closed subset satisfying (31). The shading dimension of X is defined as the smallest r such that there exist unit vectors $v_1, \ldots, v_{r+1}$ such that the intersection

$$\bigcap_{i=1}^{r+1} \mathrm{Shade}_{v_i}(X) = X \tag{32}$$

equals X. Equivalently, $r + 1$ is the minimal number of projectors (placed at infinity) needed to illuminate the space $\mathbb{R}^n - X$.

EXAMPLE 24.5. Let $X \subset \mathbb{R}^n$ be a finite set $X = \{p_1, \ldots, p_m\}$ then its shading dimension is 1. Indeed, choose a generic unit vector $u \in S^{n-1}$ such that no line through p_i and p_j has direction u. Then u and $-u$ are two directions such that the intersection of their shades equals X. Any line in $\mathbb{R}^n$ in the direction of u intersects X in at most one point and hence the unit vectors u and $-u$ illuminate the whole complement $\mathbb{R}^n - X$.

THEOREM 24.6. *If* $X \subset \mathbb{R}^n$ *is closed subset satisfying* (31) *then for the topological complexity of the complement* $\mathbf{TC}(\mathbb{R}^n - X)$ *one has*

$$\mathbf{TC}(\mathbb{R}^n - X) \leq 2r + 1$$

where r *is the shading dimension of* X. *Moreover, using the discussion of Section* 20 *one obtains an explicit motion planning algorithm in* $\mathbb{R}^n - X$ *with* $\leq 2r + 1$ *local rules.*

Proof. It follows from the results described above, since

$$\mathbb{R}^n - X = \bigcup_{i=1}^{r+1} (\mathbb{R}^n - \mathrm{Shade}_{v_i}(X))$$

and each term $\mathbb{R}^n - \mathrm{Shade}_{v_i}(X)$ is contractible. □

25. Illuminating the complement of the braid arrangement

Consider n particles in $\mathbb{R}^m$ which are disjoint from each other. In this case the obstacle set $X \in \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m = \mathbb{R}^{mn}$ is

$$X = \bigcup_{i<j} H_{ij}$$

where H_{ij} is the linear subspace $z_i = z_j$, i.e., the particle number i collides with the particle number j.

Let $e \in \mathbb{R}^m$ be a fixed unit vector. Let

$$v = (0, e, 2e, \dots, (n-1)e) \in \mathbb{R}^{nm}.$$

We will consider the shade of X in the direction of v. First note that v is generic, i.e., it is not parallel to any of the subspaces H_{ij}.

Let $z \in \mathbb{R}^m$ be a point. Its *tail* is defined as the set $T(z) = \{z - \lambda e; \lambda \geq 0\}$.

LEMMA 25.1. *The shade* $\mathrm{Shade}_v(X) \subset \mathbb{R}^{nm}$ *coincides with the set of all configurations* $(z_1, z_2, \dots, z_n) \in \mathbb{R}^{nm}$, *where* $z_i \in \mathbb{R}^m$, *such that*

$$z_i \in T(z_j) \quad \textit{for at least one pair } i < j. \tag{33}$$

Proof. Consider a configuration $(z_1, z_2, \dots, z_n) \in X$. Assume that it lies in H_{ij}, i.e., $z_i = z_j$ where $i < j$. Then the current configuration of the shade is $(z'_1, z'_2, \dots, z'_n)$ where $z'_i = z_i + (i-1)\lambda e$. We see that $z'_j - z'_i = (j-i)\lambda e$ which means that z'_i lies in the tail of z'_j, i.e., $z'_i \in T(z'_j)$.

Conversely, suppose now that we are given a configuration $z = (z_1, z_2, \dots, z_n)$ such that $z_i \in T(z_j)$ for some $i < j$. Then $z_j = z_i + (j-i)\lambda e$ for some $\lambda > 0$. We see that the configuration $z' = (z'_1, z'_2, \dots, z'_n)$ where $z'_r = z_r - (r-1)\lambda e$, lies in H_{ij} and hence z lies in the shade of z' in the direction of vector v. □

Note that the complement of the described set in the configuration space is indeed contractible (in accordance with Lemma 24.3). Since we have

$$z_i \notin T(z_j) \quad \text{for all } i < j,$$

one may first move the point z_n far enough in the direction of vector $-e$, there will be no obstacles. Then one moves the point z_{n-1} again in the direction of $-e$ also far, but closer than z_n. And so on: each next point is moved not that far so that the points after the motion lie in different slices of $\mathbb{R}^m$ (no interactions).

26. A quadratic motion planning algorithm in $F(\mathbb{R}^m, n)$

Combining the general recipe for constructing motion planning algorithms described in Section 20 with Theorem 24.6 and the results of Section 25, one may

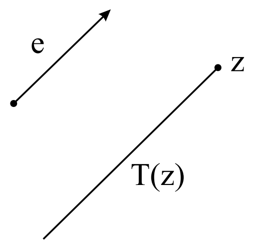

Figure 17.

construct an explicit motion planning algorithm with $\leq n^2$ local rules where n is the number of particles. In this section we briefly explain how such algorithm can be built.

Fix distinct unit vectors $e_1, e_2, \dots, e_N \in \mathbb{R}^m$, where

$$N = \frac{n(n-1)}{2} + 1.$$

Then for any configuration $z = (z_1, \dots, z_n) \in F(\mathbb{R}^m, n)$ where $z_i \neq z_j$, $z_i \in \mathbb{R}^m$, there exists $1 \leq r \leq N$ such that the vector e_r (one of the N fixed unit vectors) is distinct from all vectors

$$\frac{z_i - z_j}{|z_i - z_j|}, \quad \text{for all } i < j.$$

Therefore the configuration z lies in the complement of the shade

$$\mathbb{R}^{nm} - \text{Shade}_{e_r}(X).$$

Hence N contractible sets $\mathbb{R}^{nm} - \text{Shade}_{e_r}(X)$, where $r = 1, \dots, N$, cover the complement $\mathbb{R}^{nm} - X$. By the construction of Section 20 this leads to a motion planning algorithm with

$$2N - 1 = n^2 - n + 1$$

local rules.

27. Configuration spaces of graphs

Here we will discuss the configuration spaces $F(\Gamma, n)$ where Γ is a connected graph. These spaces were studied by Ghrist (2001), Ghrist and Koditschek (2002) and Abrams (2002); see also Gal (2001), Świątkowski (2001). To illustrate the importance of these configuration spaces for robotics one may mention the control problems where a number of automated guided vehicles (AGV) have to move along a network of floor wires. The motion of the vehicles must be safe: it should be organized so that the collisions do not occur. If n is the number of AGV then the natural configuration space of this problem is $F(\Gamma, n)$ where Γ is a graph.

The first question to ask is whether the configuration space $F(\Gamma, n)$ is connected. Clearly $F(\Gamma, n)$ is disconnected if $\Gamma = [0, 1]$ is a closed interval (and $n \geq 2$) or if $\Gamma = S^1$ is the circle and $n \geq 3$. These are the only examples of this kind as the following simple lemma claims:

LEMMA 27.1. *Let Γ be a connected finite graph having at least one essential vertex. Then the configuration space $F(\Gamma, n)$ is connected.*

An essential vertex is a vertex which is incident to 3 or more edges.

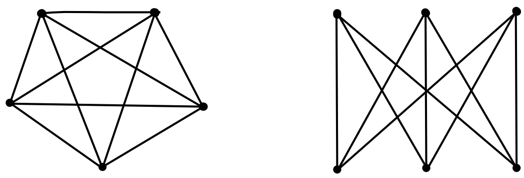

Figure 18.

THEOREM 27.2. *Let* Γ *be a connected graph having an essential vertex. Then the topological complexity of* $F(\Gamma, n)$ *satisfies*

$$\mathbf{TC}(F(\Gamma, n)) \leq 2m(\Gamma) + 1, \tag{34}$$

where $m(\Gamma)$ *denotes the number of essential vertices in* Γ.

A proof can be found in Farber (2005).

THEOREM 27.3. *Let* Γ *be a tree having an essential vertex. Let* n *be an integer satisfying* $n \geq 2m(\Gamma)$ *where* $m(\Gamma)$ *denotes the number of essential vertices of* Γ. *In the case* $n = 2$ *we will additionally assume that the tree* Γ *is not homeomorphic to the letter* Y *viewed as a subset of the plane* $\mathbb{R}^2$. *Then the upper bound* (34) *is exact, i.e.,*

$$\mathbf{TC}(F(\Gamma, n)) = 2m(\Gamma) + 1. \tag{35}$$

Farber (2005) contains a sketch of the proof and also an explicit description of a motion planning algorithm in $F(\Gamma, n)$ (assuming that Γ is a tree) having precisely $2m(\Gamma) + 1$ domains of continuity.

If Γ is homeomorphic to the letter Y then $m(\Gamma) = 1$ and $F(\Gamma, 2)$ is homotopy equivalent to the circle S^1. Hence in this case $\mathbf{TC}(F(\Gamma, 2)) = 2$. The equality (35) fails in this case.

For any tree Γ one has $\mathbf{TC}(F(\Gamma, 2)) = 3$ assuming that Γ is not homeomorphic to the letter Y. This shows that the assumption $n \geq 2m(\Gamma)$ of Theorem 27.3 cannot be removed: if Γ is a tree with $m(\Gamma) \geq 2$ then the inequality above would give $\mathbf{TC}(F(\Gamma, 2)) = 2m(\Gamma) + 1 \geq 5$.

Here are more examples. For the graphs K_5 and $K_{3,3}$ (Figure 18) one has

$$\mathbf{TC}(F(K_5, 2)) = \mathbf{TC}(F(K_{3,3}, 2)) = 5. \tag{36}$$

In these examples the equality (35) is violated.

28. Motion planning in projective spaces

Next we consider the problem of computing the topological complexity of the real projective spaces. We will follow Farber et al. (2003) which shows that the

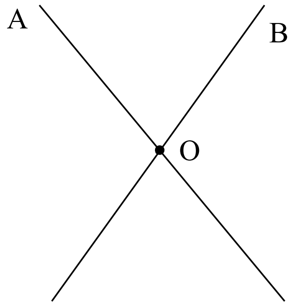

Figure 19.

problem of computing the number $\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$ is equivalent to a classical problem of manifold topology which asks what is the minimal dimension of the Euclidean space N such that there exists an immersion $\mathbb{R}\mathbf{P}^n \to \mathbb{R}^N$. The immersion problem for the real projective spaces was studied by many people and a variety of important results was obtained. However at the moment the immersion dimension of $\mathbb{R}\mathbf{P}^n$ as a function of n is not known. We refer to a recent survey (Davis, 1993).

The problem of finding motion planning algorithms in the projective space $\mathbb{R}\mathbf{P}^n$ can be viewed as an elementary problem of topological robotics. Indeed, points of $\mathbb{R}\mathbf{P}^n$ represent lines through the origin in the Euclidean space $\mathbb{R}^{n+1}$ and hence a motion planning algorithm in $\mathbb{R}\mathbf{P}^n$ should describe how a given line A in $\mathbb{R}^{n+1}$ should be moved to another prescribed position B.

Lines through the origin in $\mathbb{R}^3$ may represent metallic bars fixed at the fixed point by a revolving joint; this situation is common in the practical robotics.

If the angle between the lines A and B is acute then one may rotate A toward B in the two-dimensional plane spanned by A and B such that A sweeps the acute angle. Hence the problem reduces immediately to the special case when the lines A and B are orthogonal. In this case, if the intention is to use simple rotations, one needs a continuous choice of the direction of rotation in the plane spanned by A and B.

Note that the Lusternik – Schnirelmann category of the real projective spaces is well known and easy to compute: $\mathrm{cat}(\mathbb{R}\mathbf{P}^n) = n+1$. Using the general properties of the topological complexity mentioned above we may write

$$n + 1 \leq \mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq 2n + 1.$$

We shall see below (see Corollary 30.4) that in fact $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq 2n$ for all n; the equality holds if n is a power of 2.

The answer in the complex case is much simpler:

LEMMA 28.1. $\mathbf{TC}(\mathbb{C}\mathbf{P}^n) = 2n + 1$. *More generally, for any simply connected symplectic manifold M one has*

$$\mathbf{TC}(M) = \dim M + 1.$$

Proof. Let $u \in H^2(M)$ be the class of the symplectic form. We have a zero-divisor $u \otimes 1 - 1 \otimes u$ satisfying

$$(u \otimes 1 - 1 \otimes u)^{2n} = (-1)^n \binom{2n}{n} u^n \otimes u^n$$

where $2n = \dim M$. The cohomological lower bound gives $\mathbf{TC}(M) \geq 2n + 1$. The cohomological upper bound of Farber (2004) (using the assumption that M is simply connected) gives the opposite inequality $\mathbf{TC}(M) \leq 2n + 1$. □

THEOREM 28.2. *If* $n \geq 2^{r-1}$ *then* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \geq 2^r$.

Proof. Let $\alpha \in H^1(\mathbb{R}\mathbf{P}^n; \mathbb{Z}_2)$ be the generator. The class $\alpha \times 1 + 1 \times \alpha$ is a zero-divisor. Consider the power

$$(\alpha \times 1 + 1 \times \alpha)^{2^r - 1}.$$

Assuming that $2^{r-1} \leq n < 2^r$ it contains the nonzero term

$$\binom{2^r - 1}{n} \alpha^k \otimes \alpha^n$$

where $k = 2^r - 1 - n < n$. Applying the cohomological lower bound the result follows. □

29. Nonsingular maps

The main result concerning $\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$ (see Theorem 29.2) uses the following classical notion:

DEFINITION 29.1. A continuous map

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k \tag{37}$$

is called nonsingular if:

(a) $f(\lambda u, \mu v) = \lambda\mu f(u, v)$ for all $u, v \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{R}$, and

(b) $f(u, v) = 0$ implies that either $u = 0$, or $v = 0$.

In the mathematical literature there exist several variations of the notion of a nonsingular map. We refer to Lam (1967) and Milgram (1967) where nonsingular maps (of a different type) were used to construct immersions of real projective spaces into the Euclidean space.

Problem. Given n find the smallest k such that there exists a nonsingular map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$.

Let us show that for any n there exists a nonsingular map $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n-1}$. Fix a sequence $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}\colon \mathbb{R}^n \to \mathbb{R}$ of linear functionals such that any n of them are linearly independent. For $u, v \in \mathbb{R}^n$ the value $f(u, v) \in \mathbb{R}^{2n-1}$ is defined as the vector whose jth coordinate equals the product $\alpha_j(u)\alpha_j(v)$, where $j = 1, 2, \ldots, 2n-1$. If $u \neq 0$ then at least n among the numbers $\alpha_1(u), \ldots, \alpha_{2n-1}(u)$ are nonzero. Hence if $u \neq 0$ and $v \neq 0$ there exists j such that $\alpha_j(u)\alpha_j(v) \neq 0$ and thus $f(u, v) \neq 0 \in \mathbb{R}^{2n-1}$.

Remarks.

1. For $k < n$ there exist no nonsingular maps $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ (as follows from the Borsuk – Ulam theorem).
2. For $n = 1, 2, 4, 8$ there exist nonsingular maps $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ having an additional property that for any $u \in \mathbb{R}^n$, $u \neq 0$ the first coordinate of $f(u, u)$ is positive.
   These maps use the multiplication of the real numbers, the complex numbers, the quaternions, and the Cayley numbers, correspondingly.
3. For n distinct from 1, 2, 4, 8 there exist no nonsingular maps $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (as follows from the famous theorem of J. F. Adams).

Here is the main theorem of Farber et al. (2003):

THEOREM 29.2. *The number* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$ *coincides with the smallest integer* k *such that there exists a nonsingular map* $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$.

We refer to Farber et al. (2003) for the proof. Here we will only explain (following Farber et al., 2003) how one uses the nonsingular maps to construct motion planning algorithms.

PROPOSITION 29.3. *If there exists a nonsingular map* $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ *with* $n+1 < k$ *then* $\mathbb{R}\mathbf{P}^n$ *admits a motion planner with k local rules, i.e.,* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq k$.

Proof. Let $\Phi\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a scalar continuous map such that $\phi(\lambda u, \mu v) = \lambda\mu\phi(u, v)$ for all $u, v \in V$ and $\lambda, \mu \in \mathbb{R}$. Let $U_\phi \subset \mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$ denote the set of all pairs (A, B) of lines in $\mathbb{R}^{n+1}$ such that $A \neq B$ and $\phi(u, v) \neq 0$ for some points $u \in A$ and $v \in B$. It is clear that U_ϕ is open.

There exists a continuous map s defined on U_ϕ with values in the space of continuous paths in the projective space $\mathbb{R}\mathbf{P}^n$ such that for any pair $(A, B) \in U_\phi$ the path $s(A, B)(t)$, $t \in [0, 1]$, starts at A and ends at B. One may find unit vectors $u \in A$ and $v \in B$ such that $\phi(u, v) > 0$. Such pair u, v is not unique: instead of u, v we may take $-u$, $-v$. Note that both pairs u, v and $-u$, $-v$ determine the same orientation of the plane spanned by A, B. The desired map s consists in rotating A toward B in this plane, in the positive direction determined by the orientation.

Assume now additionally that $\phi\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \to \mathbb{R}$ is *positive* in the following sense: for any $u \in \mathbb{R}^{n+1}$, $u \neq 0$, one has $\phi(u,u) > 0$. Then instead of U_ϕ we may take a slightly larger set $U'_\phi \subset \mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$, which is defined as the set of all pairs of lines (A, B) in $\mathbb{R}^{n+1}$ such that $\phi(u, v) \neq 0$ for some $u \in A$ and $v \in B$. Now all pairs of lines of the form (A, A) belong to U'_ϕ. For $A \neq B$ the path from A to B is defined as above (rotating A toward B in the plane, spanned by A and B, in the positive direction determined by the orientation), and for $A = B$ we choose the constant path at A. Then continuity is not violated.

A vector-valued nonsingular map $f\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$ determines k scalar maps $\phi_1, \dots, \phi_k\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ (the coordinates) and the described above neighborhoods U_{ϕ_i} cover the product $\mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$ minus the diagonal. Since $n + 1 < k$ one may replace the initial nonsingular map by such an f that for any $u \in \mathbb{R}^{n+1}$, $u \neq 0$, the first coordinate $\phi_1(u, u)$ of $f(u, u)$ is positive. Now, the open sets $U'_{\phi_1}, U_{\phi_2}, \dots, U_{\phi_k}$ cover $\mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$. We have described explicit motion planning strategies over each of these sets. Therefore $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq k$.

30. TC($\mathbb{R}\mathbf{P}^n$) and the immersion problem

THEOREM 30.1. *For any* $n \neq 1, 3, 7$ *the number* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$ *equals the smallest* k *such that the projective space* $\mathbb{R}\mathbf{P}^n$ *admits an immersion into* $\mathbb{R}^{k-1}$.

The proof (see Farber et al., 2003) uses Theorem 29.2 and the following theorem of Adem et al. (1972):

THEOREM 30.2. *There exists an immersion* $\mathbb{R}\mathbf{P}^n \to \mathbb{R}^k$ (*where* $k > n$) *if and only if there exists a nonsingular map* $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$.

We will give here a direct construction of a motion planning algorithm in $\mathbb{R}\mathbf{P}^n$ starting from an immersion $\mathbb{R}\mathbf{P}^n \to \mathbb{R}^k$.

THEOREM 30.3. *Suppose that the projective space* $\mathbb{R}\mathbf{P}^n$ *can be immersed into* $\mathbb{R}^k$. *Then* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq k + 1$.

Proof. Imagine $\mathbb{R}\mathbf{P}^n$ being immersed into $\mathbb{R}^k$. Fix a frame in $\mathbb{R}^k$ and extend it, by parallel translation, to a continuous field of frames. Projecting orthogonally onto $\mathbb{R}\mathbf{P}^n$, we find k continuous tangent vector fields $v_1, v_2, \dots, v_k$ on $\mathbb{R}\mathbf{P}^n$ such that the vectors $v_i(p)$ (where $i = 1, 2, \dots, k$) span the tangent space $T_p(\mathbb{R}\mathbf{P}^n)$ for any $p \in \mathbb{R}\mathbf{P}^n$.

A nonzero tangent vector v to the projective space $\mathbb{R}\mathbf{P}^n$ at a point A (which we understand as a line in $\mathbb{R}^{n+1}$) determines a line $\hat{v}$ in $\mathbb{R}^{n+1}$, which is orthogonal to A, i.e., $\hat{v} \perp A$. The vector v also determines an orientation of the two-dimensional plane spanned by the lines A and $\hat{v}$, see Figure 20.

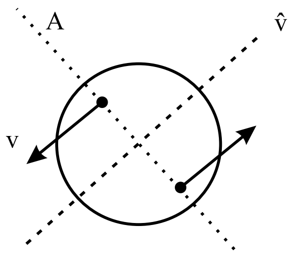

Figure 20.

For $i = 1, 2, \ldots, k$ let $U_i \subset \mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$ denote the open set of all pairs of lines (A, B) in $\mathbb{R}^{n+1}$ such that the vector $v_i(A)$ is nonzero and the line B makes an acute angle with the line $\widehat{v_i(A)}$. Let $U_0 \subset \mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$ denote the set of pairs of lines (A, B) in $\mathbb{R}^{n+1}$ making an acute angle.

The sets $U_0, U_1, \ldots, U_k$ cover $\mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$. Indeed, given a pair (A, B), there exist indices $1 \leq i_1 < \cdots < i_n \leq k$ such that the vectors $v_{i_r}(A)$, where $r = 1, \ldots, n$, span the tangent space $T_A(\mathbb{R}\mathbf{P}^n)$. Then the lines

$$A, \widehat{v_{i_1}(A)}, \ldots, \widehat{v_{i_n}(A)}$$

span the Euclidean space $\mathbb{R}^{n+1}$ and therefore the line B makes an acute angle with one of these lines. Hence, (A, B) belongs to one of the sets $U_0, U_{i_1}, \ldots, U_{i_k}$.

We may describe a continuous motion planning strategy over each set U_i, where $i = 0, 1, \ldots, k$. First define it over U_0. Given a pair $(A, B) \in U_0$, rotate A toward B with constant velocity in the two-dimensional plane spanned by A and B so that A sweeps the acute angle. This defines a continuous motion planning section $s_0 \colon U_0 \to P(\mathbb{R}\mathbf{P}^n)$. The continuous motion planning strategy $s_i \colon U_i \to P(\mathbb{R}\mathbf{P}^n)$, where $i = 1, 2, \ldots, k$, is a composition of two motions: first we rotate line A toward the line $\widehat{v_i(A)}$ in the in the 2-dimensional plane spanned by A and $\widehat{v_i(A)}$ in the direction determined by the orientation of this plane (see above). On the second step rotate the line $\widehat{v_i(A)}$ toward B along the acute angle similarly to the action of s_0. □

COROLLARY 30.4. *One has* $\mathbf{TC}(\mathbb{R}\mathbf{P}^n) \leq 2n$.

Proof. The case $n = 1$ is trivial. For $n > 1$ by the Whitney immersion theorem there exists an immersion $\mathbb{R}\mathbf{P}^n \to \mathbb{R}^{2n-1}$. The result now follows from Theorem 30.3. □

Below is the table of the values $\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$ for $n \leq 23$, see Farber et al. (2003). It is obtained by combining the results mentioned above with the information on the immersion problem available in the literature.

TABLE I.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$	2	4	4	8	8	8	8	16	16	17	17	19
n	13	14	15	16	17	18	19	20	21	22	23	
$\mathbf{TC}(\mathbb{R}\mathbf{P}^n)$	23	23	23	32	32	33	33	35	39	39	39	

As explained in Farber et al. (2003) explicit motion planning algorithms in $\mathbb{R}\mathbf{P}^n$ with $n \leq 7$ could be constructed using the multiplication of the complex numbers, the quaternions, and the Cayley numbers.

31. Some open problems

Finally we mention several open problems concerning the homotopy invariant $\mathbf{TC}(X)$.

1. *Rational version of* $\mathbf{TC}(X)$. It can be "formally" defined as $\mathbf{TC}(X_{\mathbb{Q}})$. One should be able to express this number in terms of Sullivan's minimal model. This result may give stronger (more sophisticated) lower bounds than the cohomological lower bound mentioned above.
   The rational version of the LS category was introduced by Felix and Halperin (1982).
2. *Symmetric motion planning*. One may decide to impose on the motion planning algorithms $s: X \times X \to PX$ two additional (quite natural) conditions: (a) The path $s(A, A)$ is a constant path at point A; (b) For $A \neq B$ one has $s(A, B)(t) = s(B, A)(1 - t)$. In other words, the motion from B to A goes along the same route as the motion from A to B but in the reverse order.
   The appropriate numerical invariant $\mathbf{TC}^S(X)$ measuring the topological complexity is defined as *one plus the Schwartz genus of the fibration*

   $$(P'X)/\mathbb{Z}_2 \to (X \times X - \Delta)/Z_2.$$

   Here $P'X$ is the set of paths $\gamma: [0, 1] \to X$ with $\gamma(0) \neq \gamma(1)$.
   It has the following properties: (A) $\mathbf{TC}^S(X) \geq \mathbf{TC}(X)$; (B) In some examples $\mathbf{TC}^S(X) > \mathbf{TC}(X)$; (C) The number $\mathbf{TC}^S(X)$ is not a homotopy invariant of X.

   Problem. Find a cohomological lower bound for $\mathbf{TC}^S(X)$.

3. *Motion planning in aspherical spaces*. The problem is to compute $\mathbf{TC}(X)$ in the case when the polyhedron X is aspherical, i.e., $\pi_i(X) = 0$ for all $i > 1$. The

homotopy type of an aspherical space X depends only on the fundamental group $\pi = \pi_1(X)$. Hence in this case the number $\mathbf{TC}(X)$ depends only on the group π viewed as a discrete group. One should be able to express the number $\mathbf{TC}(X)$ in terms of the algebraic properties of the group $\pi_1(X)$.
A similar question for the Lusternik – Schnirelmann category was solved by Eilenberg and Ganea (1957). Their theorem states: *If X is aspherical then*

$$\mathrm{cat}(X) - 1 = \dim \pi = \mathrm{geom}\dim \pi \tag{38}$$

except 3 *special low-dimensional cases*. Here $\dim \pi$ is the least n such that $H^q(\pi; A) = 0$ for any module A and for any $q > n$. The symbol $\mathrm{geom}\dim \pi$ denotes the smallest dimension of a $K(\pi, 1)$-complex.

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