Seminar Sophus Lie
1 (1991) 83–91

Lie Theory of Differential Equations and Computer Algebra

Günter Czichowski

Introduction

The aim of this contribution is to show the possibilities for solving ordinary differential equations with algorithmic methods using Sophus Lie's ideas and computer means. Our material is related especially to Lie's work on transformations and differential equations—essential ideas are already contained in his first paper on transformation groups [5]—and to his article on differential invariants [6]. Very good modern surveys on such questions as are discussed here and on related problems are found in [8,9].

Lie's first intentions were to create a theory for solving differential equations with means of group theory in analogy with the Galois theory for algebraic equations. With respect to typical elements of Galois theory—fields, groups, automorphisms and relations betweeen them—this concept is realized today in the so-called Picard-Vessiot theory for linear ordinary differential equations. Those of Lie's methods which are used today in systematic investigations of differential equations are based on symmetries. We will discuss here these methods and inspect them for the presence of algorithmic elements.

Symmetries

A *symmetry* of a differential equation is a transformation which transforms solutions into solutions. For the application of analytical methods it is useful to narrow this notion down as follows:

Definition. The elements of a connected Lie group G of diffeomorphisms of $\mathbb{R}^2$ which transform solutions of a differential equation

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

into solutions are called *symmetries of this equation.* Alternatively, the infinitesimal generators of the Lie algebra $\mathfrak{g}$ of such a group G are also called *symmetries* of the differential equation.

This is usually expressed also by saying "the differential equation is invariant with respect to G" or "the differential equation admits G".

Remark. In this sense, Lie methods use only connected symmetry groups. For instance, the differential equation $y'' = (xy' - y)^3$ is invariant with respect to $\mathrm{Sl}(2)$ and its natural action on $\mathbb{R}^2$. But the reflection $y \mapsto -y$ is a symmetry in the original sense of the word this is not covered by Lie methods.

In order to make the notion of symmetry practical it is necessary to describe first the action of point transformations on the derivatives ocurring in a differential equation. Let $G = \{T_t\}$ denote a (local) one-parameter group of diffeomorphisms on R^2, defined as follows:

$$T_t(x, y) = \big(\Phi(x, y, t), \Psi(x, y, t)\big).$$

If for fixed t we set $(\overline{x}, \overline{y}) = \big(\Phi(x, y, t), \Psi(x, y, t)\big)$ and if $x \mapsto y = y(x)$ is a function, then, under suitable conditions on domains of definition etc., there is a function $\overline{x} \mapsto \overline{y}(\overline{x})$ such that the relation

$$\big(\overline{x}, \overline{y}(\overline{x})\big) = \big(\Phi(x, y(x), t), \Psi(x, y(x), t)\big) = T_t\big(x, y(x)\big)$$

is satisfied. The transformation T_t now produces a transformation $T_t^{(1)}$ on $\mathbb{R}^3$ such that $T_t^{(1)}\big(x, y(x), y'(x)\big) = \big(\overline{x}, \overline{y}(\overline{x}), \overline{y}'(\overline{x})\big)$ and so on for all higher derivatives. We obtain transformations $T_t^{(n)}$ realizing in this fashion the assignments

$$y \mapsto \overline{y},\ y' \mapsto \overline{y}', \dots,\ y^{(n)} \mapsto \overline{y}^{(n)}.$$

Now the elements of the (local) one-parameter group $\{T_t\}$ are symmetries of $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ iff the following implication holds:

$$y^{(n)} = F(x, y', \dots, y^{(n-1)}) \Longrightarrow \overline{y}^{(n)} = F(\overline{x}, \overline{y}, \overline{y}', \dots, \overline{y}^{(n-1)}).$$

Since the transformation formulas for the derivatives $y', y'', \dots, y^{(n)}$ are rather complicated, it is more convenient to describe the transformation groups $T_t^{(n)}$ by their generators. If, in the following, ∂_v means the differentiation with respect to v, and if the vector field generating T_t is denoted by

$$\partial = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

then the extension to the level of derivatives leads to a sequence

$$\begin{aligned} \partial &= \xi(x, y)\partial_x + \eta(x, y)\partial_y, \\ \partial' &= \partial + \eta'(x, y, y')\partial_{y'}, \\ &\vdots \\ \partial^{(n)} &= \partial^{(n-1)} + \eta^{(n)}(x, y, \dots, y^{(n)})\partial_{y^{(n)}} \end{aligned}$$

of generators, and this extension procedure is given recursively by

$$\eta^{(k+1)} = \frac{d\eta^{(k)}}{dx} - y^{(k+1)}\frac{d\xi}{dx}.$$

Its implementation requires no more than a lot of simple calculations, especially differentiations.

There are two criteria for symmetries going back to LIE [7]:

Criterion 1. *The generator $\partial = \xi\partial_x + \eta\partial_y$ is a symmetry of*

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \quad \text{iff} \quad \partial^{(n)}(y^{(n)} - F)_{|y^{(n)}=F} \equiv 0.$$

The second criterion is formulated in terms of the differential operator

$$D = \partial_x + y'\partial_y + \ldots + y^{(n-1)}\partial_{y^{(n-2)}} + F\partial_{y^{(n-1)}},$$

which is associated with the given differential equation as the total differentiation with respect to x by means of the differential equation. The kernel of D is the space of first integrals of the differential equation.

Criterion 2 *The generator $\partial = \xi\partial_x + \eta\partial_y$ is a symmetry of*

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \quad \textit{iff} \quad [\partial^{(n-1)}, D] = \lambda D$$

where λ is a certain function depending on $x, y, y', \ldots, y^{(n-1)}$.

Criterion 1 is useful for the computation of symmetries: For $n > 1$ the corresponding identity contains the free variables $y', \ldots, y^{(n-1)}$, which allow us to split this identity into a system of linear partial differential equations for the unknown functions ξ and η.

From Criterion 2 it follows easily that the symmetries (as generators) form a Lie algebra and that the kernel of D (the space of first integrals) is invariant under the action of symmetries.

Algorithmic procedure

The effect of Lie methods applied to ordinary differential equations can be characterized as follows:

An ordinary differential equation with symmetries can be reduced to lower order equations and quadratures.

For a given equation $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$ one has to work in the following steps:

(1) Assuming a symmetry as $\partial = \xi\partial_x + \eta\partial_y$ with unknown functions ξ and η, one has to extend the generator ∂ to generators $\partial^{(n)}$ and to form the identity $\partial^{(n)}(y^{(n)} - F)\big|_{y^{(n)}=F} \equiv 0$, corresponding to Criterion 1.

(2) By splitting the above identity with respect to the free variables $y', \ldots, y^{(n-1)}$ (for $n > 1$) one gets the system of determining equations. These are linear homogeneous partial differential equations for ξ and η.

(3) Symmetries are obtained by simplifying and solving the determining system.

(4) For the Lie algebra L of symmetries one has to determine the type of isomorphy, the canonical form and a corresponding base.

(5) Depending on this informations one gets a solution procedure, i. e., there are additional first order linear partial differential equations to determine first integrals. This is equivalent to the reduction of the differential equation mentioned above.

Let us now refer to the steps of this algorithm and its realization. The steps (1) and (2) pose no problems and can be implemented with computer algebra means, too. The essential problem in finding symmetries is to solve the system of determining equations in step (3). At first glance this problem seems not to be easier than the solution of the original differential equation. But this impression is disproved by experience: As many concrete examples show, the system of determining equations leads by a simplification procedure (which is an analogue to the Gröbner base algorithm for algebraic equations [11]) to simple equations. We will formulate this as

Hypothesis. *Let S be the determining system of an ordinary differential equation. Then one can derive from S, by means of differentiations and combinations of equations only, an equation for only one function, which is in fact an ordinary linear homogeneous differential equation.* ■

Here we give two examples:

(1) The determining system of $y'' = \frac{yy'}{x} + {y'}^2$ is:

$$\begin{aligned}
\xi_{yy} + \xi_y &= 0, \\
2\xi_{xy} + 2\frac{y}{x}\xi_y + -\eta_{yy} + \eta_y &= 0, \\
-\xi_{xx} - \frac{y}{x}\xi_x + \frac{y}{x^2}\xi + 2\eta_{xy} - 2\eta_x - \frac{1}{x}\eta &= 0, \\
\eta_{xx} - \frac{y}{x}\eta_x &= 0.
\end{aligned}$$

The first and the last equation are obviously ordinary differential equations.

(2) For the equation

$$y'' = \frac{(1+{y'}^2)^{3/2} - {y'}^2 - 1}{y}$$

we get the determining equations

$$\begin{aligned}
y\xi_x - 2y\eta_y + \eta &= 0, \\
\xi_y + \eta_x &= 0, \\
-2y\xi_x + y\eta_y + \eta &= 0, \\
y\xi_{yy} - \xi_y &= 0, \\
-2y^2\xi_{xy} + y^2\eta_{yy} + y\eta_y - \eta &= 0, \\
2y\xi_x + y^2\eta_{xx} - y\eta_y - \eta &= 0.
\end{aligned}$$

Here the fourth equation is an ordinary differential equation for ξ.

At this point it is clear, that Lie methods in the algorithmic sense are not self-consistent: As a consequence of the superposition principle, for linear homogeneuos equations, the determination of all symmetries is equivalent to solving the equation itself. This is true in the case of arbitrary first order equations, too. There are only special cases with enough known symmetries, for which work Lie methods effectively. In this sense, if the hypothesis is true, then the solving of linear homogeneous equations and first order equations (for instance, quadratures) is the key for effective applications of Lie methods. Hence in these cases, with respect to the algorithmic point of view and computer means, it is necessary to apply other methods. We will quote here only some results, which express the progress in this topic and are based on methods resembling GALOIS theory:

The problem of integrating elementary functions has been proved to be algorithmic [1]. The theory is based on old ideas of LIOUVILLE. There are computer implementations in partial cases, too. For certain classes of linear homogeneous equations there exist algorithms [4, 10] which lead to Liouvillean solutions, if such solutions exist.

Use of symmetries for solving procedures

Let $\partial = \xi\partial_x + \eta\partial_y$ be a symmetry of the n-th order equation

$$y^{(n)} = F(x, y', \ldots, y^{(n-1)}).$$

Then we can introduce new variables $\overline{x} = \Phi(x,y)$, $\overline{y} = \Psi(x,y)$, where $\overline{y}$ is considered as function depending on $\overline{x}$, in such a way that $\partial = \partial_{\overline{y}}$. Then the differential equation with respect to the new variables attains the form $\overline{y}^{(n)} = \overline{F}(\overline{x}, \overline{y}', \ldots, \overline{y}^{(n-1)})$ ($\overline{y}$ itself does not occur). Hence we have to solve an $(n-1)$-th order differential equation for $\overline{y}'$, and $\overline{y}$ is given then by a quadrature.

In the general case, this means that if there is a Lie algebra L of symmetries, the methods given by LIE are split with respect to various isomorphy types and canonical forms of L. We will try to explain a concept for solving a differential equation by first integrals in this way.

The canonical form of a Lie algebra L of generators is the equivalence class with respect to point transformations which contains L. For instance, every one-dimensional Lie algebra L can be transformed by point transformations $(x,y) \mapsto (\overline{x}, \overline{y})$ into $L = \langle \partial_{\overline{y}} \rangle$. Therefore $L = \langle \partial_y \rangle$ is also called *the canonical form of a one-dimensional Lie algebra.* For a two-dimensional abelian Lie algebra there are two canonical forms: $L = \langle \partial_x, \partial_y \rangle$ or $L = \langle \partial_x, y\partial_x \rangle$. (In fact these concrete Lie algebras stand for the whole classes).

The determination of the canonical forms is a kind of representation theory, which was worked out for lower dimensional Lie algebras by LIE. The concept of solving or reducing a differential equation with symmetries is based on the following proposition.

Proposition. *Let $y^{(n)} = F(x, y', \dots, y^{(n-1)})$ be a differential equation with the Lie algebra of symmetries L. Let further I denote the space of first integrals, i. e., of functions*

$$u = u(x, y, y', \dots, y^{(n-1)}) \quad \textit{with} \quad D(u) = 0,$$

where $D = \partial_x + y'\partial_y + F\partial_{y^{(n-1)}}$. Then the mapping $\Phi\colon \partial \to \partial^{(n-1)}\big|_I$ is a Lie algebra monomorphism.

The proof is not hard if one uses Criterion 2 and the relation

$$[\partial_1, \partial_2]^{(k)} = [\partial_1{}^{(k)}, \partial_2{}^{(k)}].$$

Hence, if I is regarded, with respect to n independent first integrals, as a space of functions depending on n variables, there is a canonical form for L with respect to transformations only among first integrals. We will denote this form as FI-form of L. If the FI-form is known, one gets additional equations for first integrals. Let us illustrate this fact by some examples of second order equations. Our goal is to obtain two independent first integrals $u = u(x, y, p)$, $v = v(x, y, p)$ (here p stands for y' as in the following, too).

(1) If $y'' = F(x, y, p)$ has one symmetry ∂, the canonical FI-form can be obtained as $\partial'|_I = \partial_v$. But this means that there are first integrals u, v forming a base of I, which satisfy

$$\begin{aligned} D(u) &= 0, & D(v) &= 0, \\ \partial'(u) &= 0, & \partial'(v) &= 1. \end{aligned}$$

This is a first order system of linear partial differential equations for u and v, which is equivalent to ordinary first equations and quadratures and can be regarded as the final result of our procedure.

(2) The equation

$$y'' = \frac{(1+p^2)^{3/2} + 2(1+p^2)(xp - y)}{(1 + x^2 + y^2)}$$

has $\mathbf{so}(3)$-symmetry. The corresponding generators are

$$\begin{aligned} \partial_1 &= y\partial_x - x\partial_y, \\ \partial_2 &= \frac{(1 + x^2 - y^2)}{2}\partial_x + xy\partial_y, \\ \partial_3 &= xy\partial_x + \frac{(1 - x^2 + y^2)}{2}\partial_y. \end{aligned}$$

The group is SU(2) acting by Möbius transformations on $\mathbb{C}$. There is only one canonical form on R^2. Hence one gets the canonical FI-form analogously to the above equations. This implies the existence of first integrals satisfying

$$\begin{array}{ll} \partial_1'(u) = v, & \partial_1'(v) = -u, \\ \partial_2'(u) = \frac{(1+u^2-v^2)}{2}, & \partial_2'(v) = uv, \\ \partial_3'(u) = uv, & \partial_3'(v) = \frac{(1-u^2+v^2)}{2}, \\ D(u) = 0, & D(v) = 0. \end{array}$$

Elimination of the derivatives of u, v leads to two algebraic equations for u, v. With the help of computer algebra means we can then eliminate p ad get the general solution as a polynomial equation in x, y, u, v of very large volume. (u, v can then be considered as constants for every solution). In the simplest case $u = 0, v = 0$ one gets $y^2(-x^8 - 4x^6y^2 + x^6 - 6x^4y^4 + 3x^4y^2 + 4x^4 - 4x^2y^6 + 3x^2y^4 + 8x^2y^2 + x^2 - y^8 + y^6 + 4y^4 + y^2 - 1) = 0$.

(3) The Lie algebra $\mathbf{sl}(2)$ has 3 canonical forms with respect to real point transformations:

$$\begin{aligned} \partial_1 &= \partial_x, \\ \partial_2 &= x\partial_x + y\partial_y, \\ \partial_3 &= \frac{(x^2 + \varepsilon y^2)}{2}\partial_x + xy\partial_y, \quad (\varepsilon = 0, 1, -1). \end{aligned}$$

$\varepsilon = 0$ corresponds to the linear action of Sl(2) on $\mathbb{R}^2$,
$\varepsilon = -1$ to the action of Sl(2) by Möbius transformations on $\mathbb{C}$,
$\varepsilon = 1$ to the action by simultaneous Möbius transformations on R^2.

With respect to complex transformations the cases $\varepsilon = +1, -1$ coincide. Hence for a given differential equation with $\mathbf{sl}(2)$ symmetry one must know again the canonical FI-form, i. e., the corresponding ε value. For instance, in the case $y'' = (xp - y)^3$ (invariance with respect to the linear action of Sl(2) on R^2) the canonical FI-form is that with $\varepsilon = 1$. I. e., if the symmetries are given by $L = \langle \partial_1, \partial_2, \partial_3 \rangle$ with $[\partial_1, \partial_2] = \partial_1$, $[\partial_1, \partial_3] = \partial_2$, $[\partial_2, \partial_3] = \partial_3$, then there are first integrals u, v satisfying

$$\begin{array}{ll} \partial_1'(u) = 1, & \partial_1'(v) = 0, \\ \partial_2'(u) = u, & \partial_2'(v) = v, \\ \partial_3'(u) = \frac{u^2+v^2}{2}, & \partial_3'(v) = uv, \\ D(u) = 0, & D(v) = 0. \end{array}$$

Concrete generators in the case of linear action of Sl(2) are

$$\partial_1 = \frac{-y}{\sqrt{2}}\partial_x, \quad \partial_2 = \frac{x}{2}\partial_x - \frac{y}{2}\partial_y, \quad \partial_3 = \frac{x}{\sqrt{2}}\partial_y.$$

With these generators, an elimination procedure analogous to that in the previous example leads us via computer algebra to the following general solution (u, v can now be regarded as constants):

$2x^2y^2v^3 - x^4y^2v^2 - y^2v^4 + 2x^3y^3uv^2 - x^2y^4u^2v^2 + 2xyuv^2 - 2x^3yuv - x^2y^4v^4 + x^2y^2u^2v - y^2u^2v^2 - 2x^2v^2 + x^4v + v^3 = 0$.

Analogously, for every second order ordinary differential equation with 2 or 3 known symmetries, there is a procedure which is based on the canonical FI-Form of its Lie algebra and which reduces the differential equation to quadratures or to a system of algebraic equations.

Invariants

The problem of obtaining the information about the canonical FI-form can be solved by computing special cases corresponding to the various canonical forms and by subsequently using differential invariants as labels.

A *differential invariant (with respect to point transformations) for a differential equation* $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ is a function Ω depending on the arguments of F (regarded as independent variables) and on the partial derivatives of F (as dependent variables), which is invariant under the action of point transformations (absolute invariants) or which is multiplied by a factor being a certain function (relative invariants). The order of Ω is the order of the highest derivative of F occuring in Ω.

Examples for second order equations $y'' = F(x, y, p)$, $(p = y')$ are the following relative invariants I_1, I_2 [3]:

$$I_1 = F_{pppp},$$
$$I_2 = D^2(F_{pp}) - 4D(F_{py}) - F_pD(F_{pp}) + 4F_pF_{py} - 3F_yF_{pp} + 6F_{yy}.$$

Here D denotes again the operator $\partial_x + p\partial_y + F\partial_p$.

We will illustrate the use of such invariants by the following

Proposition. *Let* $y'' = F(x, y, p)$ *be a differential equation with* $\mathbf{sl}(2)$*-symmetry. Then the canonical FI-form is given by* $\varepsilon = 1$ *iff* $I_2 = 0$*, and by* $\varepsilon = 0$ *iff* $I_2 \neq 0$.

Remark. For the other cases of second order equations with symmetries there is only one canical FI-form for every type of isomorphy.

There are old ideas of LIE's [6] to compute such invariants directly by big systems of first order linear partial differential equations. The author has followed this path and proved with the help of computer algebra the following result.

Proposition. *For the general second order equation* $y'' = F(x, y, p)$ *the functions* AI_1, AI_2 *below are absolute invariants with respect to point transformations. There are no nontrivial absolute invariants of order less than six.*

$$AI_1 = I_1^{-11} I_2 (6{F_{p5}}^2 - 5I_1F_{p6})^4,$$
$$AI_2 = \frac{(6{F_{p5}}^2 - 5I_1F_{p6})^2}{(25I_1^2F_{p7} + 84{F_{p5}}^3 - 105F_{p5}F_{p6})}.$$

(Here F_{p5} *means* F_{ppppp} *and so on).*

References

[1] Bronstein, M., J. H. Davenport and B. M. Trager, *Symbolic Integration is algorithmic!* Computers & Mathematics 1989.

[2] Czichowski, G., *Behandlung von Differentialgleichungen mit Lie-Theorie und Computer*, Mitt. d. MGdDDR, Heft **3-4** (1988), 3–20.

[3] Kamran, N., *Contributions to the study of the equivalence problem of Elie Cartan and its applications to partial and ordinary differential equations*, Preprint 1988.

[4] Kovacic, J. J., *An Algorithm for Solving Second Order Linear Homogeneous Differential Equations*, J. Symb. Comp. **2** (1986), 3–43.

[5] Lie, S., *Über Gruppen von Transformationen*, Göttinger Nachrichten Nr. 22 (1874), 529–542.

[6] —, *Über Differentialinvarianten*, Math. Ann. **24** (1884), 537–578.

[7] —, „Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen", Verlag B. G. Teubner, Leipzig 1891.

[8] Schwarz, F., *Symmetries of differential equations: From Sophus Lie to computer algebra*, SIAM Review **30/3** (1988).

[9] Singer, M. F., *Formal Solutions of Differential Equations*, J. Symb. Comp. **10** (1990), 59—94.

[10] —, *Liouvillean solutions of n th order linear differential equations* Amer. J. Math. **103** (1981).

[11] Wolf, T., *An analytic algorithm for decoupling and integrating systems of nonlinear partial differential equations*, J. Comp. Physics **60/3** (1985).

Fachrichtungen Mathematik/Informatik
Ernst Moritz Arndt-Universität
Jahnstrasse 15a
O-2200 Greifswald

Received February 18, 1991