Higher Gauge Theory
and Elliptic Cohomology

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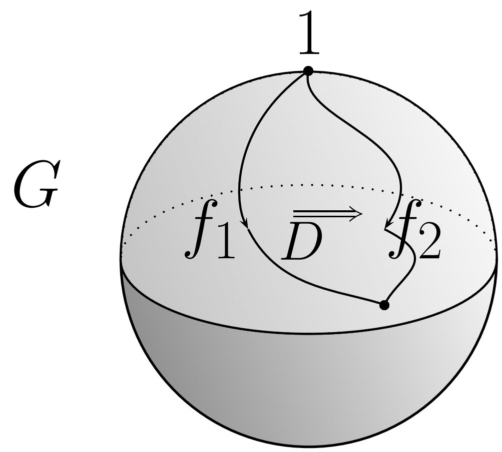


For more see: http://math.ucr.edu/home/baez/abel/

Categorification

sets $\rightsquigarrow$ **categories**
functions $\rightsquigarrow$ **functors**
equations $\rightsquigarrow$ **natural isomorphisms**

Categorification 'boosts the dimension' by one:

$$\begin{tikzcd} \bullet \arrow[r, bend left=15] & \bullet \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \bullet \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

In **strict** categorification we keep equations as equations. This is evil... but today we'll do it whenever it doesn't cause trouble, just to save time.

Higher Gauge Theory

$$
\begin{aligned}
\textbf{groups} &\rightsquigarrow \textbf{2-groups}\\
\textbf{Lie algebras} &\rightsquigarrow \textbf{Lie 2-algebras}\\
\textbf{bundles} &\rightsquigarrow \textbf{2-bundles}\\
\textbf{connections} &\rightsquigarrow \textbf{2-connections}
\end{aligned}
$$

Connections describe parallel transport for particles.
2-Connections describe parallel transport for strings!

$$
\begin{tikzcd}
\bullet \arrow[r, bend left] & \bullet & \rightsquigarrow & \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, phantom, "\Downarrow"]
\end{tikzcd}
$$

We should even go beyond $n = 2$... but not today.

Fix a simply-connected compact simple Lie group G. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\mathfrak{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\mathrm{String}_k(G)$.
- The geometric realization of the nerve of $\mathrm{String}_k(G)$ is a topological group, $|\mathrm{String}_k(G)|$.
- Principal $\mathrm{String}_k(G)$-2-bundles are the same as $|\mathrm{String}_k(G)|$-bundles.
- For $k = 1$, $|\mathrm{String}_k(G)|$ is G with its 3rd homotopy group made trivial.
- Connections on $\mathrm{String}_k(G)$-2-bundles are related to Stolz and Teichner's elliptic objects!

2-Groups

A **strict 2-group** is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

The objects in a 2-group look like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

The morphisms look like this:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "f"] \end{tikzcd}$$

We can multiply objects: $\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "g'"] & \bullet \end{tikzcd}$

multiply morphisms: $\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "f_1 \Downarrow"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "f_2 \Downarrow"] & \bullet \end{tikzcd}$

and compose morphisms: $\begin{tikzcd} \bullet \arrow[r, bend left=70, "g"] \arrow[r, "g'"] \arrow[r, bend right=70, "g''"'] \arrow[r, phantom, shift left=3, "\Downarrow f"] \arrow[r, phantom, shift right=3, "\Downarrow f'"] & \bullet \end{tikzcd}$

All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=70] \arrow[r, bend left=20] \arrow[r, bend right=20] \arrow[r, bend right=70] \arrow[r, phantom, shift left=4ex, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, shift right=4ex, "\Downarrow"] & \bullet \end{tikzcd}$$

Finally, the **interchange law** holds, meaning

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow"] & \bullet \end{tikzcd}$$

is well-defined.

Mac Lane and Whitehead first introduced 2-groups in the disguise of 'crossed modules':

$$G_0 \xleftarrow{\partial} G_1$$

Here G_0 and G_1 are groups, and G_0 acts on G_1 in a manner compatible with the differential ∂.

To get a crossed module from a 2-group, just let G_0 be the group of objects:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd}$$

and G_1 be the group of morphisms starting at 1. The differential ∂ is defined as follows:

$$\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=50, "1"{name=U}] \arrow[r, bend right=50, "\partial(f)"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"] \end{tikzcd}$$

Lie 2-Algebras

A **strict Lie 2-algebra** is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

A strict Lie 2-algebra can be viewed as an 'infinitesimal crossed module':

$$\mathfrak{g}_0 \xleftarrow{\ \partial\ } \mathfrak{g}_1$$

Here $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are Lie algebras, and $\mathfrak{g}_0$ acts as derivations of $\mathfrak{g}_1$ in a manner compatible with the differential ∂.

Theorem (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group G of isomorphism classes of objects,
- the abelian group A of endomorphisms of any object,
- an action of G on A,
- an element of $H^3(G, A)$.

Theorem (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $\mathfrak{g}$ of isomorphism classes of objects,
- the vector space $\mathfrak{a}$ of endomorphisms of any object,
- a representation of $\mathfrak{g}$ on $\mathfrak{a}$,
- an element of $H^3(\mathfrak{g}, \mathfrak{a})$.

Suppose G is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

Corollary. For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\mathfrak{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

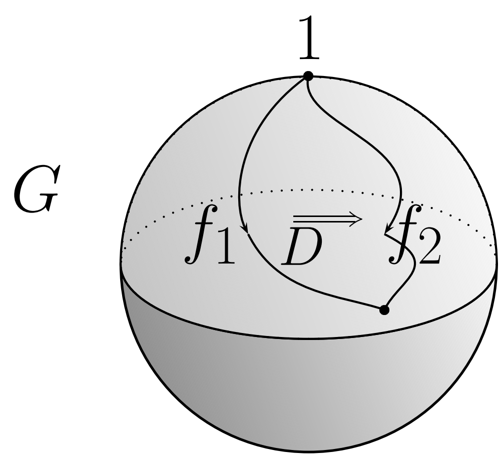
Every Lie 2-algebra with these properties is equivalent to $\mathfrak{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$.

Theorem. For any $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ is the Lie 2-algebra of an infinite-dimensional Lie 2-group $\mathrm{String}_k(G)$.

An object of $\mathrm{String}_k(G)$ is a smooth path

$$f\colon [0, 2\pi] \to G$$

starting at the identity. A morphism from f_1 to f_2 is an equivalence class of pairs (D, α) where D is a disk going from f_1 to f_2 and $\alpha \in \mathrm{U}(1)$:



Any two such pairs (D_1, α_1) and (D_2, α_2) have a 3-ball B whose boundary is $D_1 \cup D_2$. The pairs are equivalent when

$$\exp\left(2\pi i k \int_B \nu\right) = \alpha_2/\alpha_1$$

where ν is the left-invariant closed 3-form on G with

$$\nu(x, y, z) = \langle [x, y], z \rangle$$

and $\langle \cdot, \cdot \rangle$ is the smallest invariant inner product on $\mathfrak{g}$ such that ν gives an integral cohomology class.

Theorem. The morphisms in $\mathrm{String}_k(G)$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

For any topological 2-group $\mathcal{G}$ there is a topological group $|\mathcal{G}|$ built as the geometric realization of the nerve of $\mathcal{G}$.

Theorem. For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups

$$1 \longrightarrow K(\mathbb{Z},2) \longrightarrow |\mathrm{String}_k(G)| \xrightarrow{p} G \longrightarrow 1$$

where p is a fibration. Using this we can show:

$$\pi_2(|\mathrm{String}_k(G)|) \cong \mathbb{Z}/k\mathbb{Z}$$

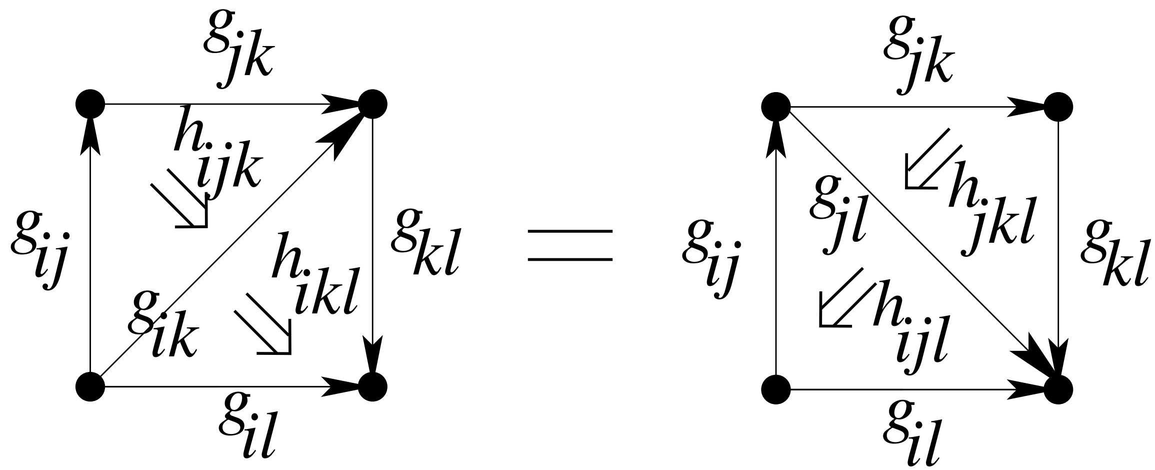
When $k = \pm 1$, $|\mathrm{String}_k(G)|$ is the topological group formed by making the 3rd homotopy group of G trivial.

2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space X, we can define a **principal $\mathcal{G}$-2-bundle over** X to consist of:

- an open cover U_i of X,
- continuous maps
$$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$
satisfying $g_{ii} = 1$, and
- continuous maps
$$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$
with
$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

satisfying the **nonabelian 2-cocycle condition:**



on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

There's a natural notion of 'equivalence' for 2-bundles over X, since they form a 2-category.

Theorem. For any topological 2-group $\mathcal{G}$ and paracompact Hausdorff space X, there is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over X,
- isomorphism classes of principal $|\mathcal{G}|$-bundles over X,
- homotopy classes of maps $f\colon X \to B|\mathcal{G}|$.

So, $B|\mathcal{G}|$ is the classifying space for $\mathcal{G}$-2-bundles.

2-Connections — Quick and Dirty

Let $\mathcal{G}$ be a Lie 2-group, P the trivial principal $\mathcal{G}$-2-bundle over some smooth manifold X. A **2-connection** on P assigns holonomies to paths in X:

$$
\mathrm{hol}\colon\ \begin{tikzcd} x \arrow[r, bend left, "\gamma"] & y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, bend left, "\mathrm{hol}(\gamma)"] & \bullet \end{tikzcd} \in \mathrm{Ob}(\mathcal{G})
$$

and surfaces going between paths:

$$
\mathrm{hol}\colon\ \begin{tikzcd} x \arrow[r, bend left=50, "\gamma"] \arrow[r, bend right=50, "\eta"'] \arrow[r, phantom, "\Downarrow\Sigma"] & y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{hol}(\gamma)"] \arrow[r, bend right=50, "\mathrm{hol}(\eta)"'] \arrow[r, phantom, "\Downarrow\mathrm{hol}(\Sigma)"] & \bullet \end{tikzcd} \in \mathrm{Mor}(\mathcal{G})
$$

in a manner preserving all 3 forms of composition:

$$
\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r, bend left] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \bullet \end{tikzcd}
$$

Theorem. Let

$$\mathfrak{g}_0 \xleftarrow{\partial} \mathfrak{g}_1$$

be the infinitesimal crossed module obtained by differentiating the crossed module

$$G_0 \xleftarrow{\partial} G_1$$

corresponding to $\mathcal{G}$. Then there is a 1-1 correspondence between 2-connections on $P \to X$ and pairs consisting of:

- a $\mathfrak{g}_0$-valued 1-form A on X
- a $\mathfrak{g}_1$-valued 2-form B on X

satisfying the **fake flatness** condition:

$$dA + \frac{1}{2}[A, A] + \partial B = 0$$

All this generalizes to nontrivial 2-bundles... but not today. Instead, let's say a bit about how 2-connections relate to Stolz and Teichner's work on elliptic objects.

For any group G, any principal G-bundle $P \to X$ gives an associated vector bundle

$$E = P \times_G \mathbb{C}^G$$

where $\mathbb{C}^G$ is the set of functions

$$f \colon G \to \mathbb{C}$$

Any connection on P induces a connection on E.

For a full-fledged Lie group we should not really use $\mathbb{C}^G$. We should pay more attention to analysis.

Stolz and Teichner's work may involve a *categorification* of this idea!

For any 2-group $\mathcal{G}$, any principal $\mathcal{G}$-2-bundle $P \to X$ gives an associated '2-vector 2-bundle'

$$E = P \times_{\mathcal{G}} \mathrm{Vect}^{\mathcal{G}}$$

where $\mathrm{Vect}^{\mathcal{G}}$ is the category of functors

$$f \colon \mathcal{G} \to \mathrm{Vect}$$

Any 2-connection on P induces a 2-connection on E.

For a full-fledged Lie 2-group we should not really use $\mathrm{Vect}^{\mathcal{G}}$. We should pay more attention to analysis. But not today...

The fun starts when we set $\mathcal{G} = \mathrm{String}_k(G)$.

Theorem. Let $\mathcal{G} = \mathrm{String}_k(G)$. Then $\mathrm{Vect}^{\mathcal{G}}$ is the category of nicely PG-graded $\widehat{\Omega_k G}$-modules, where a grading is **nice** if

$$\deg(\tilde{f} v) = f \deg(v)$$

for all $\tilde{f} \in \widehat{\Omega_k G}$, hence $f \in \Omega G \subset PG$.

Many structures seen by Stolz and Teichner arise when we:

- Describe how a 2-connection on a principal $\mathcal{G}$-2-bundle $P \to X$ induces a 2-connection on the associated bundle
$$E = P \times_{\mathcal{G}} \mathrm{Vect}^{\mathcal{G}}$$

Locally, such a 2-connection assigns functors

$$\mathrm{Vect}^{\mathcal{G}} \to \mathrm{Vect}^{\mathcal{G}}$$

to paths in X, and natural transformations to surfaces going between paths.

- Using the theorem, describe $\mathrm{Vect}^{\mathcal{G}}$ as the category of modules of a certain algebra

$$A = \mathbb{C}^{PG} \rtimes \mathbb{C}[\widehat{\Omega_k G}]$$

Then, describe the above functors as 'tensoring with A-A bimodules', and the natural transformations as 'tensoring with bimodule homomorphisms'.

- Get a kind of 2-connection that assigns A-A bimodules to paths in X, and bimodule homomorphisms to surfaces going between paths.

A pretty picture — but the puzzle pieces don't all fit yet:

- We should pay more attention to analysis, to get von Neumann algebras into the game.
- Stolz and Teichner use a different representation of $\mathcal{G} = \mathrm{String}_k(G)$, based on a different algebra A — see the work of Urs Schreiber. How is this related?
- Most importantly: their sort of 2d parallel transport uses surfaces *with conformal structure.*

The big challenge is to categorify further, and study n-connections for higher n. A whole new world awaits us!