1. What is elliptic cohomology?

Definition 1. *For a ring R, an R-valued genus on a class of closed manifolds is a function φ that assigns to each manifold M an element $\varphi(M) \in R$ such that*

(i) $\varphi(M_1 \coprod M_2) = \varphi(M_1) + \varphi(M_2)$
(ii) $\varphi(M_1 \times M_2) = \varphi(M_1)\varphi(M_2)$
(iii) $\varphi(M) = 0$ *if M is a boundary.*

Equivalently, φ is a homomorphism from the appropriate cobordism ring Ω to R.

Definition 2. *A 1-dimensional formal group law over R is a power series $F(x, y) \in R[[x, y]]$ satisfying*

(i) $F(x, 0) = F(0, x) = x$.
(ii) $F(y, x) = F(x, y)$
(iii) $F(x, F(y, z)) = F(F(x, y), z)$.

A theorem of Quillen says that in the complex case (where $\Omega = MU_*$, the complex cobordism ring), φ is a equivalent to a 1-dimensional formal group law over R. It is also known that the functor

$$X \mapsto MU_*(X) \otimes_\varphi R$$

is a homology theory if φ satisfies certain conditions spelled out in Landweber's Exact Functor Theorem.

Now suppose E is an elliptic curve defined over R. It is a 1-dimensional algebraic group, and choosing a local paramater at the identity leads to a formal group law $\widehat{E}$, the formal completion of E. Thus we can apply the machinery above and get an R-valued genus.

For example, the *Jacobi quartic*, defined by the equation

$$y^2 = 1 - 2\delta x^2 + \epsilon x^4,$$

is an elliptic curve over the ring

$$R = \mathbf{Z}[1/2, \delta, \epsilon].$$

The resulting formal group law is the power series expansion of

$$F(x, y) = \frac{x\sqrt{1 - 2\delta y^2 + \epsilon y^4} + y\sqrt{1 - 2\delta x^2 + \epsilon x^4}}{1 - \epsilon x^2 y^2};$$

this calculation is originally due to Euler. The resulting genus is known to satisfy Landweber's conditions, and this leads to one definition of elliptic cohomology.

2. What does "chromatic" mean?

The stable homotopy category localized a prime p can be studied via a series of increasingly complicated localization functors L_n for $n \geq 0$, which detect "v_n-periodic" phenomena.

- L_0 is rationalization. Rational stable homotopy theory is very well understood. It detects only the 0-stem in the stable homotopy groups of spheres.
- L_1 is localization with respect to K-theory. It detects the image of J and the α family in the stable homotopy groups of spheres. The Lichtenbaum-Quillen conjecture is a statement about L_1 of algebraic K-theory.
- L_2 is equivalent to localization with respect to elliptic cohomology as defined above. It detects the β family in the stable homotopy groups of spheres. Davis' nonimmersion theorm for real projective spaces was proved using related methods. The theory of topological modular forms of Hopkins *et al* is a refinement of elliptic cohomolgy.
- For $n > 2$ there is no comparable geometric definition of L_n, which can only be constructed by less illuminating algebraic methods related to BP-theory. It detects higher Greek letter families in the stable homotopy groups of spheres. The nth Morava K-theory is closely related to it.

A key to understanding the algebraic underpinnings of the chromatic point of view is the following.

Definition 3. *Let F be 1-dimensional formal group law over a field k of characteristic p. For a positive integer m, the m series is defined inductively by*

$$[m]_F(x) = F(x, [m-1]_F(x))$$

where $[1]_F(x) = x$. The p-series is either 0 or has the form

$$[p]_F(x) = ax^{p^n} + \cdots$$

for some nonzero $a \in k$. The height of F is the integer n. If $[p]_F(x) = 0$ (which happens when $F(x, y) = x + y$), the height is defined to be ∞.

Here are some examples:

- The multiplicative formal group law (which is associated with K-theory) has height 1.
- The formal group law associated with an elliptic curve is known to have height at most 2.
- v_n-periodic phenomena (the nth layer in the chroatic tower) are related to formal group laws of height n.

2. What is a higher chromatic analog?

QUESTION: How can we attach formal group laws of height > 2 to geometric objects (such as algebraic curves) and use them get insight into cohomology theories that go deeper into the chromatic tower?

PROGRAM:

- Let C be a curve of genus g over some ring R.
- Its Jacobian $J(C)$ is an abelian variety of dimension g.
- $J(C)$ has a formal completion $\widehat{J}(C)$ which is a g-dimensional formal group law.
- If $\widehat{J}(C)$ has a 1-dimensional summand, then Quillen's theorem gives us a genus associated with the curve C.

Theorem 4. *Let $C(p,e)$ be the Artin-Schreier curve over $\mathbf{F}_p$ defined by the affine equation*

$$y^e = x^p - x \qquad \text{where } e = p^e - 1.$$

(Assume that $e > 1$ when $p = 2$.) Then its Jacobian has a 1-dimensional formal summand of height $(p-1)e$.

Conjecture 5. *Let $\tilde{C}(p,e)$ be the curve over over $\mathbf{Z}_p[u_1, \ldots, u_{(p-1)e-1}]$ defined by*

$$y^e = x^p - x + \sum_{i=0}^{(p-1)e-2} u_{i+1} x^{p-1-[i/e]} y^{p^{e-1}-p^{i-[i/e]e}}.$$

Then its Jacobian has a formal 1-dimensional summand isomorphic to the Lubin-Tate lifting of the formal group law of height $(p-1)e$.

Properties of $C(p,e)$:

- Its genus is $(p-1)(d-1)/2$.
- It has an action by the group
$$G = \mathbf{F}_p \rtimes \mu_{(p-1)e}$$
given by
$$(x,y) \mapsto (\zeta^d x + a, \zeta y)$$
for $a \in \mathbf{F}_p$ and $\zeta \in \mu_{(p-1)e}$. This group is a maximal finite subgroup of the $(p-1)$eth Morava stablizer group, and it acts appropriately on the 1-dimensional formal summand.
- The case $e = 1$ was studied by Gorbunov-Mahowald.

Examples:

- $C(2,2)$ and $C(3,1)$ are elliptic curves whose formal group laws have height 2.
- $C(2,3)$ has genus 3 and a 1-dimensional formal summand of height 3.
- $C(2,4)$ and $C(3,2)$ each has genus 7 and a 1-dimensional formal summand of height 4.

Remarks

- This result was known to and cited by Manin in 1963. Most of what is needed for the proof can be found in Katz's 1979 Bombay Colloquium paper and in Koblitz' Hanoi notes.
- The proof rests on the determination of the zeta function of the curve by Davenport-Hasse in 1934, and on some properties of Gauss sums proved by Stickelberger in 1890. The method leads to complete determination of $\widehat{J}(C(p,e))$.
- Let $\mathbf{G}_n$ denote the extension of the Morava stabilizer group S_n by the Galois group C_n. Given a finite subgroup $G \subset \mathbf{G}_n$, Hopkins-Miller can construct a "homotopy fixed point spectrum" E_n^{hG}. The group G from the previous page was shown by Hewett to be a maximal finite subgroup of $\mathbf{G}_n$ for $n = (p-1)f$. It acts on the 1-dimensional summand of $\widehat{J}(C(p,e))$ in the appropriate way.
- Gorbunov-Mahowald studied this curve for $e = 1$. They found a lifting of the curve to characteristic zero associated with the Lubin-Tate lift of the formal group law of height $p-1$.
- We reproved the theorem using Honda's theory of commutative formal group laws developed in the early '70s. This proof does not rely on knowledge of the zeta function and is thus a more promising approach to the lifting problem.

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