The Hairy Ball Theorem via Sperner's Lemma

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It is well known that any continuous tangent vector field on the sphere S^2 must, at some location, be zero. This result is known as the *Hairy Ball Theorem* for it can be loosely interpreted as follows:

It is impossible to comb all the hairs of a fuzzy ball so that: i) *each hair lies tangent to the surface of the ball, and* ii) *the angles of the hairs vary continuously over the surface of the ball. (By this we mean that the angle between two hairs at positions p and q say can be made arbitrarily small by choosing q sufficiently close to p.) Any attempt to accomplish this feat must produce a cowlick.*

We are assuming that *every* point of the ball's surface sprouts a hair. It is a surprise to learn that this topological result, like Brouwer's famous fixed point theorem [2, pp: 21 - 24], also follows from an application of Sperner's lemma.

1. SPERNER'S LEMMA. In 1928 Emanuel Sperner presented a simple, yet surprisingly powerful, combinatorial lemma about triangles [5]. We work with a slight generalization of his original result:

Lemma 1: *If the boundary vertices of an arbitrary triangulated polygon are labeled either "A," "B" or "C," in such a way that all the A-B edges that appear have the same orientation, then any attempt to label the interior vertices, again either "A," "B" or "C," necessarily produces at least n subtriangles fully labeled A-B-C. Here n is the number of exterior A-B edges initially scribed.*

The following constructive proof is due to Cohen [1] and Kuhn [3].

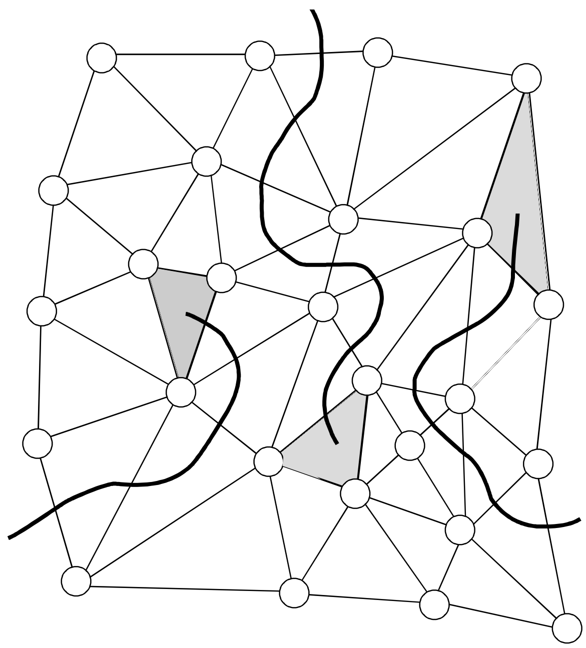

Figure 1: Paths must all end inside the palace

Proof: Imagine the triangulated polygon is the floor-plan of a palace with triangular rooms and with all the A-B edges, both inside the polygon and along its boundary, the doorways. All other edges are walls. There are n doors through which you can enter the palace from the outside. If you do so, and follow the passageway of rooms and doors as far as possible, your path must either terminate within the palace, or lead you back outside through an A-B door. The latter case is impossible, as all paths through doorways maintain the orientation of label "A" to one particular side and label "B" to the other. As one cannot enter the same room twice (no room contains three A-B doors) the n passageways must terminate in n distinct rooms. These final rooms are A-B-C triangles!■

If one does not orient the outside A-B edges appropriately one may enter an exterior door of one orientation and later exit the palace through a door of the opposite orientation. But if there is an excess of exterior doors of a particular orientation, then we can still be assured of the existence of fully labeled triangular rooms. As a generalization of Lemma 1 we therefore have:

Lemma 2: *Any labeling scheme of the vertices of a triangulated polygon,* *using the labels "A," "B" or "C," necessarily contains at least d fully labeled A-B-C subtriangles, where d is the difference in the number of exterior A-B edges of each orientation.*

Of course, our choice to focus on "A-B" doors is arbitrary.

2. THE HAIRY BALL THEOREM. We now use Lemma 2 to prove the Hairy Ball Theorem. We begin by assuming that a continuous non-zero tangent vector field on S^2 does exist, and use this supposed vector field to produce a labeling of a triangulated polygon.

By the continuity assumption there exists an open disc P on S^2 about the north pole N within which all hairs essentially point in the same direction. That is, given a prescribed value ε, any two tangent vectors selected from P have angle at most ε between them. For simplicity, we'll work with the value ε equals one degree.

Imagine the following diagram of circles drawn on the surface of the ball. These circles are mutually tangent at N and we orient them to produce a diagram that looks like the magnetic field of a dipole. One of the circles is a great circle and divides the sphere into two hemispheres.

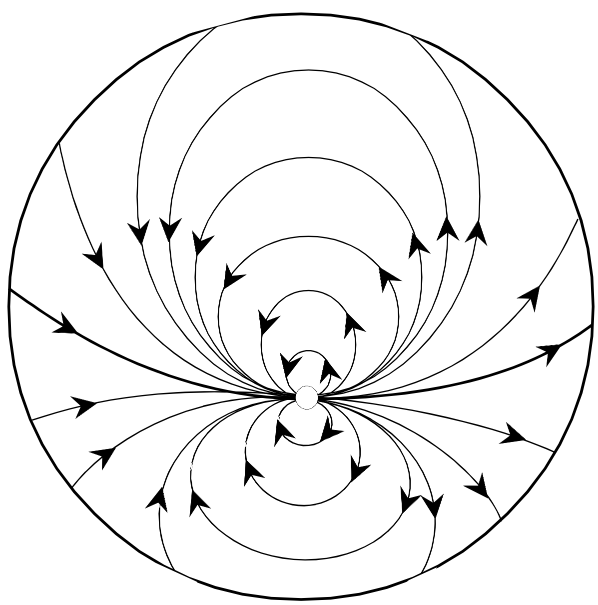

Figure 2: Circles on the sphere

Let p be a point on the sphere different from N. Then p lies on a unique oriented circle in our diagram. Let $\theta(p)$ be the angle measured in degrees counterclockwise from the direction of the hair at p to the direction of the unit tangent vector to the circle at p. Label this point "A" if $\theta(p)$ lies in the interval $[0, 120)$, "B" if $\theta(p) \in [120, 240)$, and "C" otherwise. In this way, every point on the surface of the ball, except the north pole, is given a unique label "A," "B" or "C."

Now consider the boundary ∂P of the disc P. We see that the angle θ described above undergoes two full turns if we traverse ∂P once. By continuity, it is possible to find points p_A, p_B and p_C along ∂P, and all within one hemisphere, and points q_A, q_B and q_C, on ∂P, in the opposite hemisphere, with $\theta(p_A) = \theta(q_A) = 60^o$, $\theta(p_B) = \theta(q_B) = 180^o$ and $\theta(p_C) = \theta(q_C) = 300^o$. And again, by continuity and the fact that we have chosen $\varepsilon = 1^o$, we can be sure that all the points along the boundary between p_B and p_C have labels either "B" or "C" only, all points between p_C and q_A have labels "C" and "A" only, and so on. See Figure 3.

Now triangulate $S^2 - P$, the region of the sphere outside the open disc, in any manner you care to choose, using a large number of points along the boundary of P for vertices, but including the specific points $p_{A,}p_B, p_C, q_A, q_B$ and q_C. Given that an odd number of label changes must occur when traversing ∂P from p_A to p_B, we deduce that there must be an odd number of A-B edges along this particular arc of ∂P Thus there is an excess of one "exterior" A-B edge of a particular orientation. Another of the same orientation occurs on ∂P between q_A and q_B. Thus in any triangulation of $S^2 - P$, there exist at least two fully-labeled A-B-C triangles.

Take finer and finer triangulations of $S^2 - P$, the n-th triangulation consisting of triangles with diameter no larger than $1/n$. For each of these triangulations there exist three points $x_A^{(n)}, x_B^{(n)}$ and $x_C^{(n)}$ representing the three vertices of some fully-labeled subtriangle. As the sphere is compact, a subsequence of $\{x_A^{(n)}\}$ converges to a point x^*. Moreover, this point lies outside the open disc P.

This point x^* is the limit of points on the sphere, each labeled "A" and each part of a fully labeled subtriangle. It follows that x^* is also the limit of a sequence of points labeled "B" and a sequence of points all labeled "C." Now ask: What angle does the hair at p^* make with the unit tangent to the circle through x^*? By continuity, this angle must simultaneously be in the (closed) intervals $[0, 120]$, $[120, 240]$ and $[240, 360]$, which, of course, is impossible.

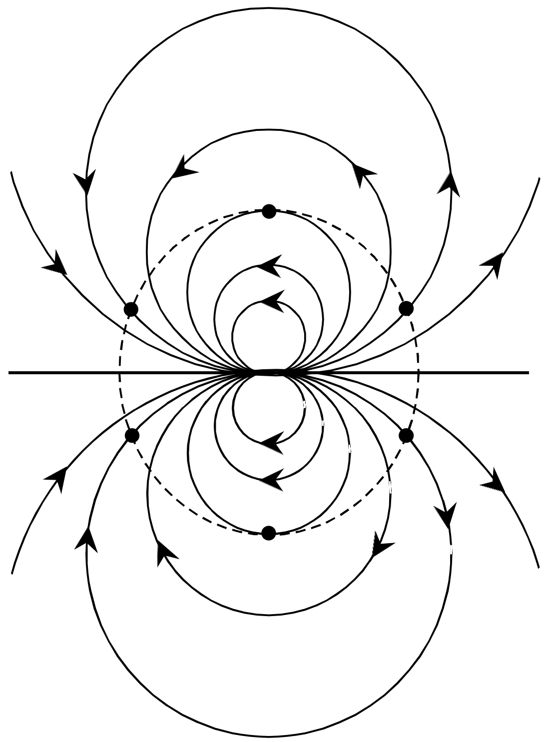

Figure 3: In this diagram all hairs (essentially) point to the right

This contradiction proves that no continuous non-zero tangent vector field can exist.

Comment: The authors would like to thank the referee for alerting them to Yuri Shashkin's treatment in [4] of additional applications of Sperner's lemma and his proof of the Hairy Ball Theorem using the idea of the degree of a map, the Fixed/Antipodal Point Theorem for a sphere, and basic homotopy theory.

REFERENCES

[1] D. I. A. Cohen, On the Sperner lemma, *J. Combin. Theory* **2** (1967) 585–587.

[2] M. Kac and S. M. Ulam, *Mathematics and Logic*, Dover, New York, 1968.

[3] H. W. Kuhn, Simplicial Approximation of Fixed Points, *Proc. Nat. Acad. Sci. U.S.A.* **61** (1968) 1238–1242.

[4] Yu. A. Shashkin, *Fixed points*, (Translated from Russian by V. Minachin), Mathematical World 2, American Math. Soc., Providence, 1991.

[5] E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, *Abh. Math. Sem. Univ. Hamburg* **6** (1928) 265–272.

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