GEOMETRY FOR PRIME ADDICTS

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Contents

Thanks to Yifeng Liu for being our advisor in this reading course.

Remark. Our goal this reading course was understand some of Scholze's recent work with perfectoid space techniques, in particular the proof of the monodromy weight conjecture (what he might call "phase 1" [1]).

This is only the texed notes of my (Catherine's) lectures. During the reading course, I got the increasing feeling that we were just studying $G_{\mathbb{Q}}$ as fast as our little legs could take us – our little legs being our knowledge of varieties (over various non-archimedian fields like $\mathbb{Q}_p$ and $F_p((t))$. So, I finished with a talk on the Grothendieck-Teichmüller group – another approach to $G_{\mathbb{Q}}$.

1. Lecture 1: The Category of Rigid Analytic Spaces: A Motivated Definition

We will take a journey through history to understand the reasoning behind the definition of rigid analytic spaces.

[1]"Scholze describes three "phases" of study, applying them to different topics in number theory. The first phase was giving a correspondence between geometry in characteristic zero and characteristic p, with the goal of proving Delignes weight-monodromy conjecture. The second phase was studying p-adic Hodge theory, and how it varies in families. The third phase discussed here is the realization of important special cases of "infinite-type rigid geometry" via perfectoid spaces." - [12]

1.1. **Quotient Models of Elliptic Curves over $\mathbb{C}$.**

Remark. $\mathbb{Q}_p$ is $\mathrm{Frac}(\mathbb{Z}_p)$, and $\mathbb{Q}_p$ is completion wrt the p-adic norm.

We will work up to the notion of rigid analytic spaces. Where did they come from? Does the formulation work for even the most basic example. We will not focus on the actual equations in the equation-based approach, for it is the most elementary way of approaching Tate's theorem.

Now, for elliptic curves over $\mathbb{C}$, we have a nice isomorphism.

$$\begin{aligned} \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) &\to E(\mathbb{C}) \\ z &\mapsto (\mathfrak{p}(z), \mathfrak{p}'(z)) \end{aligned}$$

where $\mathrm{Im}(\tau) > 0$, with the isomorphism given in terms of the Weierstraß $\mathfrak{p}$-function.

There is another way to write the curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. Let $t = e^{2\pi i \tau}$ and $\langle t \rangle = t^{\mathbb{Z}} = \{t^m | m \in \mathbb{Z}\}$. We consider the complex-analytic isomorphism:

$$\begin{aligned} \mathbb{C}/\Lambda &\to \mathbb{C}^\times / t^{\mathbb{Z}} \\ z &\mapsto e^{2\pi i z} \end{aligned}$$

This map sends the points on the usual lattice to circles, one circle with radius$k\tau$ for all $k \in \mathbb{N}$. We can think of the quotient $\mathbb{C}^\times / q^{\mathbb{Z}}$ as poking a whole in the middle of a piece of paper, this being the inside of an annulus, and then rolling up the paper.

1.2. **Quotient Models of Elliptic Curves over K.**

Let K be a field complete wrt a non-trivial non-archimedean valuation which we denote by $|\cdot|$, for example, $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$.

Question 1. Is the valuation for $F_q((t))$ the lowest degree?

If we replace $\mathbb{C}$ by K, and attempt to write $E(K)$ in the form K/Λ, we run into a serious problem, there may not be a (non-trivial) discrete subgroup.

Example 1. If $\Lambda \subset \mathbb{Q}_p$ is any nonzero subgroup, and $\lambda \in \Lambda$ then $p^n\lambda \in \Lambda$ for all $n \geq 0$, so 0 is an accumulation point of λ. It may happen that tomorrow, when you wake up, taking three steps returns you close to your starting point, and taking nine steps returns you even closer.

Example 2. $K = \mathbb{F}_q((1/t))$ is a field complete with respect to the norm $|\sum_{i \geq n} a_i (1/t)^i| = q^{-n}$. Take the completion of K with respect to the norm. Call it K'. Then, we pick an algebraic completion of this K', call it C. It is easy to see that $F_q[t]$ is discrete in C, since, for example, $F_q[t]$ intersects the unit ball only at 0 (since we will only get $(1/t)^0$ from $F_q[T]$. One can prove that the function (our analogue of the exponential) $e(x) = x \prod_{a \in F_q[T]} (1 - x/a)$ converges everywhere on C. Then fact from non-archimedean analysis implies that e is surjective, so $0 \to F_q[T] \to C \xrightarrow{e} C \to 0$ is exact.

Tate's observation was that we can use the multiplicative version of the uniformization. $K^\times$ has lots of discrete subgroups, as any $t \in K^\times$ with $0 < |t| < 1$ defines a discrete subgroup $t^{\mathbb{Z}} = \langle t \rangle$.

Theorem 3. *(Tate's theorem in 1-dimension) Taking the map in part 1, and substituting t, defines a surjective "analytic" homomorphism*

$$\phi : L^\times \to E_t(L)$$

with kernel $\langle t \rangle$ where L is any algebraic extension of K. When L is Galois over K, this homomorphism is equivariant wrt $Gal(L/K)$.

In other words, $L^\times/t^{\mathbb{Z}} \simeq E_t(L)$. All elliptic curves over K may be modeled as

$$E_t : y^2 + xy = x^3 + a_4(t)x + a_6(t)$$

for some t. So, picking the value of t completely specifies the curve (as it does in $\mathbb{C}$). All elliptic curves can not even remotely be modeled this way.

1.3. **Where do I live? Making sense of the quotient $\mathbb{G}_{m,K}^\times/\Lambda$.** We'd like to have an "analytic object" E^{an} with underlying set equal to the set of closed points of E and an isomorphism of "analytic spaces" $E^{an} \simeq \mathbb{G}_{m,K}^{an}/t^{\mathbb{Z}}$ in an appropriate geometric category.

Using X as a variable, look at the algebra of all Laurent series which are globally convergent on $K^\times$.

$$\mathcal{O}(K^\times) = \{\sum_{v\in\mathbb{N}} c_v X^v; c_v \in K, \sum_{v\in\mathbb{Z}} |c_v| r^v < \infty \text{ for all } r > 0, r \in \mathbb{R}\}$$

Viewing $\mathcal{O}(K^\times)$ as the ring of analytic functions on $K^\times$, we can construct its field of fractions $\mathcal{M}(K^\times) = \mathrm{Frac}\mathcal{O}(K^\times)$. Now we pick $t \in K^\times$ where $0 < |t| < 1$.

$$M^q(K^\times) = \{f \in \mathcal{M}(K^\times); f(tX) = f(X)\}$$

Tate saw that this field was an ELLIPTIC FUNCTION FIELD, the associated elliptic curve being $E_t(K)$.

We use a slightly different approach: we look at all power series converging on the unit ball $B_{n,K}$.

$$\mathcal{O}(B_{n,K}) = \{\sum_{\underline{v}\in\mathbb{N}} c_{\underline{v}} X^{\underline{v}}; c_{\underline{v}} \in K, \lim_{|\underline{v}|\to\infty} |c_{\underline{v}}| = 0\}$$

Remark. Note that we needed a stronger convergence condition ($\mathbb{Z}$ rather than $\mathbb{N}$) in lower dimensions.

In other words, the power series which converge on the closed n-ball. Why power series on closed balls? Because power series on closed balls admit the Gauss norm (that is the max).

$$\Big|\Big| \sum_{\underline{v}\in\mathbb{N}^n} a_{\underline{v}} Z^{\underline{v}} \Big|\Big| = \max_{\underline{v}\in\mathbb{N}^n} |a_{\underline{v}}|$$

And further, this max norm gives the ring $\mathcal{O}(K^\times)$ the structure of a Banach K-algebra. Maybe thennnn we can QUOTIENT this Banach algebra BY IDEALS. But this is not enough, this is not a space.

Question 2. Why is algebra not enough?

But, how can we phrase $\mathbb{G}_{m,K} = K^\times$ in the language of balls? Well, we can emulate $\mathbb{G}_{m,K} = K[S,T]/(ST-1)$, but we cannot ask the norms to be one, this will just give us a circle.

$$K^\times = \bigcup_{n\geq 0} \mathcal{O}(B_{n,K})/(X_1X_2 - p^n)$$

Remark. Just as the series $f \in T_n$ are viewed as analytic functions on the closed ball over K, we may view their residue classes in $T_n/\mathfrak{a}$ as analytic functions on the Zariski closed subset:

$$V(\mathfrak{a}) = \{x \in B_n(K); g(x) = 0 \text{ for all } g \in \mathfrak{a}\}$$

Let $A = T_n/\mathfrak{a}$. Then let us think of $V(\mathfrak{a})$ as being a zero set living in $Max(T_n)$, which we think of as $Max(A)$, as we do classically.

add why maximal ideals are natural to use here, what do they represent? Maps from $B \to \hat{K}^{alg}$

What shall be our definition of open set? We take the definition for finite type maps between Noetherian affine schemes being open immersions, and change algebras of the form S/I to algebras of the form T_n/I.

Definition 4. A subset $U \subseteq X$ is an pen subset of X if there is a homomorphism between K-affinoid algebras $\phi : A \to B$ with $U = \phi^*(Max(B))$, and for every homomorphism $\heartsuit : A \to C$ such that $\heartsuit^*(Max(C)) \subset U$, there is a unique homomorphism of K-affinoid algebras $\star : B \to C$ with $\heartsuit = \star \circ \phi$.

We then endow this with the Grothendieck topology. We have thus cleverly restricted the opens of interest to make our totally disconnected space have the compactness and connectness properties of schemes dear to us.

Now we have a topology on our Banach K-algebra. It is a real boy now.

That was our goal the whole time! Now we can make sense of the quotient $K^\times/\langle t\rangle$, a true blessing. So, we will give $\mathcal{O}(K^\times)$ a name, T_n.

Question 3. (Grisha) What is the internal logic of this Grothendieck topos?

Question 4. What is the relationships between Riemann surfaces and function fields? What is the analogue for higher dimensional varieties?

1.4. **Which elliptic curves over K can be constructed this way?** This is really a question of what set of questions these tools apply to, we shall see that it applies to the highly degenerate elliptic curves, and does nothing for "good" ones.

Two equivalent necessary & sufficient conditions:

- Bad reduction (via j-invariant):

  However, in the set up above, we pick $t \in K^\times$, where $0 < |t| < 1$. We can also take $|t| > 1$, (and replace all by the inverse of t below), we cannot take $|t| = 1$. Let us see how far $E_t(K)$ will take us.

  The j invariant of $E_t(X)$ is:

$$j = \frac{1}{t}(P(t))$$

where P is a power series over $\mathbb{Z}$. In K, $P(t)$ converges for $|t| < 1$, so $j \in K$. Since $|t| < 1$, then $|j| > 1$. Since $|j| > 1$, this means $j \notin R$, and thus the discriminant is not invertible. In other words, it is $0 \mod p$.

- Bad reduction (via showing we get a singular curve):
  In the non-archimedian case, $a_4(t)$ and $a_6(t)$ both converge, and further, both $a_4(t)$ and $a_6(t)$ are in the maximal ideal of the ring of integers,

$$\mathcal{O}_K := \{z \in K; |z| \leq 1\}$$
$$\mathfrak{m} := \{z \in K; |z| < 1\}$$

This means that when we reduce the curve $E_t(\mathcal{O}_K/\mathfrak{m})$, we get

$$E_t : y^2 + xy = x^3$$

Question 5. This is apparently singular, but it is smooth.

Question 6. What about p-adic exponential, what does this have to do with Tate curve?

What are the higher dimensional analogues of the Tate theorem?

To characterize the abelian varieties which admit analytic uniformization one needs three fundamental concepts: Néron models, formal schemes with their "generic fiber" functor, and the notion of analytic reduction of a rigid-analytic space. We will touch only on the briefest part of the generic fiber functor, but rest assured we will again find that only the "degenerate" abelian varieties over K will have models as rigid analytic K-spaces.

1.5. **Generic Fiber Functor.** Schemes over a valuation ring R have a generic fiber, which is a scheme over the field of fractions $K = \text{Frac}(R)$.

Grothendieck had the idea that formal schemes (of topologically finite type) over a complete valuation ring R (of dimension 1) should admit a generic fiber over the field of fractions K which, in some sense, is obtained by tensoring with K over R.

Thus, there is a functor from

$$\text{Formal R-Schemes} \to \text{Rigid K-Spaces}$$

which associates to a formal R-scheme of topologically finite type its generic fiber as rigid K-space.

We define $\tau_n = lim_k P/\mathfrak{m}^k P$, where $P = R[X_1, ..., X_n]$. Apparently this is the collection of power series convergent on the units of R.

Exercise 5. What is τ_1? Why does it have this interpretation?

Example 6. To see generic fiber we simply look at $K \otimes_R \tau_n$.

Question 7. The rational subdomains of affinoid spaces correspond to the notion of admissible formal blow ups. Apparently, over Spf A, such a blow up is the completion of a blow up on Spec A, with center in the special fiber. So, we can generalize from the base field K to quite general objects S ??

Remark. This lecture is based off of the beginnings of [1] and [2].

1.6. **On The Motivation of Huber Rings.** I emailed Huber in German to ask for his motivation for f-adic Rings, now called Huber Rings, (es scheint mir vom Himmel gekommen zu sein, ich würde gerne verstehen, wie Sie es gefunden haben!) and he kindly replied:

"Der Ausgangspunkt war, daß ich die Grothendieck-Topologie eines rigid analytischen Raums Sp(A) (also A eine klassische affinoide Algebra) und die Strukturgarbe darauf verstehen wollte. Es war klar, daßhier ein spektraler Raum im Hintergrund steht. Ich wollte diesen finden und verstehen und möglichst gut beschreiben. Nachdem mir klar war, daß man diesen topologischen Raum mit Hilfe der stetigen Bewertungen von A konstruieren kann, fragte ich mich, auf welche weiteren topologischen Ringe sich diese Konstruktion anwenden läßt. Dabei wurde klar, daß dies zum Beispiel für adische Ringe mit endlich erzeugtem Definitionsideal möglich ist. Also suchte ich nach einer Klasse topologischer Ringe, die die klassischen affinoiden Algebren und die adischen Ringe mit endlich erzeugtem Definitionsideal umfaßt (und auf die sich die Konstruktion des topologischen Raums mit Hilfe stetiger Bewertungen und die Konstruktionder Strukturgarbe verallgemeinern läßt). Hier ist doch die Klasse der f-adischen Ringe naheliegend. (Das f in f-adic bezog ich auf finitely generated ideal of definition. Ich erinnere mich an damals, nur für mich persönlich und auf Deutsch, mir fiel damals auf, f = fast (=beinahe), die Ringe sind fast adisch)."

My Translation: "The starting point was my desire to understand the Grothendieck topology on a rigid analytic space Sp(A) (that is, A is a classical affinoid algebra), and the structure sheaf over it. It was clear that there was a spectral space standing in the background. I wanted to find and understand this (space), and if possible, describe it well. After it became clear to me that one can construct this (space) by means of continuous valuations on A, I asked myself: to which other topological rings can this construction be applied? (literally: which further topological rings allow us to apply this construction). It became clear that this was possible, for example, for adic rings with a finitely generated ideal of definition. So I looked for a class of topological rings that included: the classical affinoid algebras and the adic rings with finitely generated definition ideals (and for which (1) the construction of the topological space by means of continuous evaluations and (2) the construction of the structure sheaf may both be generalized). With this in mind, the class of f-adic rings is obvious. (literally: is reclining nearby) The f in "f-adic" has two meanings: "f" as in finitely generated ideal of definition, and "f" as in "fast" (German word for "almost"). "

Question 8. Is this the universal such class of rings?

2. Lecture 2: On Spectral Theory: The Berkovich Topology on A^1_{Berk}

As we talked about last time, we are working with K an algebraically closed field which is complete wrt a non-trivial non-archimedean absolute value, $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. The topology on K induced by the given absolute value is Hausdorff, but it is also totally disconnected and not locally compact at all. What a disaster. We led up to and motivated Tate's developments as giving satisfactory theory of analytic functions on K, but it is unclear that these rigid analytic spaces give a nice notion of Laplacian, and consequently harmonic functions $(\Delta\phi) = 0$, and consequently subharmonic functions.

The main motivation, according to Yifeng, is that Tate's rigid analytic spaces did not behave well with respect to é tale cohomology. Berkovich wanted a model that behaved well wrt é tale cohomology.

Question 9. example of such misbehaving?

Remark. One motivation is a more honestly geometric alternative to Tate's category. He wanted to identify features of the geometry from information about the eigenvalues of the Laplacian, and to infer the behavior of the eigenvalues of a Riemannian manifold from knowledge of the geometry.

$$\begin{cases} \delta u + \lambda u & = 0 \\ u|_{\partial D} & = 0 \end{cases}$$

What can we infer about D knowing only the values of λ.

What is a p-adic Laplacian? If we take the philosophy of Berkovich, it will turn out to be the same as the usual graph laplacian,, let $\phi : V \to R$ be a function of vertices taking values in a ring:

$$(\Delta\phi)(v) = \sum_{w:d(v,w)=1} [\phi(v) - \phi(w)]$$

We can think of the Laplacian as measuring how much a function differs from its average (usually sum of second derivatives). We can motivate the discrete laplacian by the node modeling of a string.

The Berkovich affine line over K is a locally compact, Hausdorff, and PATH- CONNECTED topological space which contains K (with the topology given by the given absolute value as a dense subspace. WOW. They are even better then path-connected, they are uniquely path connected (any two points are joined by a unique arc).

2.1. Putting a Topology on K by Putting a Topology on the Multiplicative Semi-Norms of $K[T]$.

Let's talk about multiplicative semi-norms on a ring A. These are functions $|\cdot|_x : A \to \mathbb{R}_{\geq 0}$ satisfying the usual properties, except we take out the requirement that $|v|_x = 0$ only for nonzero v.

Remark.
(1) $|0|_x = 0, |1|_x = 1$
(2) $|fg|_x = |f|_x \cdot |g|_x$ for all $f, g \in A$
(3) $|f + g|_x \leq |f|_x + |g|_x$

Remark. The difference between this and huber rings, is that huber's adic spaces take into account valuations of higher rank.

As a set A^1_{Berk} consists of all multiplicative seminorms on the polynomial $k[T]$ which extend the usual absolute value.

Remark. To perk Grisha up, there is a taste of universality. The Berkovich topology is defined to be the weakest one for which $x \mapsto |f|_x$ is continuous for every $f \in k[T]$.

Theorem 7. *As a set, Max* $K[T_1, .., T_n] \hookrightarrow A^n_{Berk}$

Proof. (sketch) For simplicity, assume k is algebraically closed. Then Max $K[T_1, ..., T_n] \simeq K^n$. Given $a \in K^n$, we can associate a multiplicative seminorm $|\cdot|_a \in A^1_{Berk}$ by setting $|f|_a := |f(a)|$, where we think of f as a function $K^n \to K$ for all $f \in K[T_1, ..., T_n]$. □

So, we don't know all of the points in A^1_{Berk} yet, but at least we know all maximal ideals are in there. In fact, for $\mathbb{C}$, every multiplicative seminorm on $\mathbb{C}[T]$ which extends the absolute value of $\mathbb{C}$, is of the form $f \mapsto |f(z)|$ for some $z \in \mathbb{C}$.

Question 10. I don't understand why this is true, it uses a theorem called the Gelfand-Mazur theorem.

This is not at all true for K.

Let's talk about some other points in A^1_{Berk}. The multiplicative semi-norm we will use as our guiding path is the ball norm. Take $a \in K$, and $r \in \mathbb{R}_{\geq 0}$. We define the ball as $B(a,r) := \{z \in K| \ |z-a|_K \leq r\}$. We take the ball norm $|f|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|$.

Question 11. Are these different then the evaluation semi-norm on $\mathbb{C}$?

These ball norms are multiplicative by the Gauss Lemma. Consequently, each of these ball norms corresponds to a point in A^1_{Berk}.

2.2. **Non-Archimedean Balls.** It is *very* important for us to understand how balls work in the non-archimedean world to understand the Berkovich topology. It will turn out that the ball norms will be our building blocks for our other norms. Our goal is to understand why two points $|\cdot|_{B(a,r)}$ and $|\cdot|_{B(a',r')}$ are uniquely path connected in A^1_{Berk}, and what is connecting them.

To get to that goal, we will first prove that

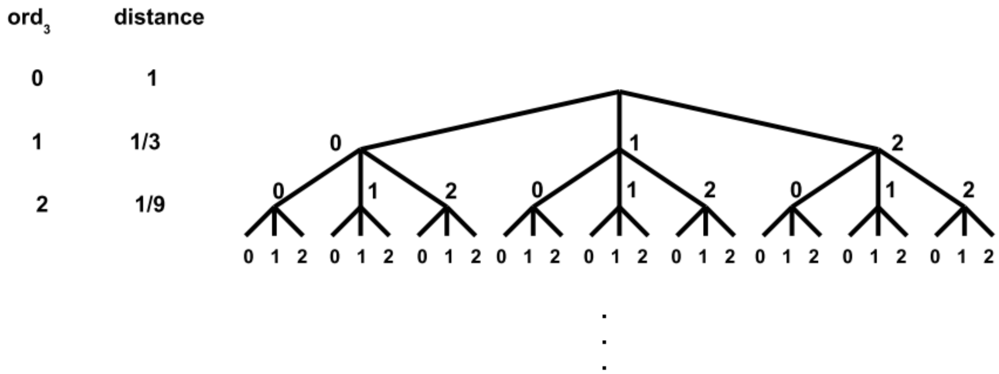
Theorem 8. *Any two balls have either empty intersection or are identified.*

We can put an equivalence relationship on the points of K: $x \sim_r y$ iff $|x-y| \leq r$. We use the non-archimedean-ness: $|x-y| = |(x-a)+(a-y)| \leq \max(|x-a|, |y-a|)$, so if we assume $|x-a| \leq r$ and $|y-a| \leq r$, we get an equivalence relation.

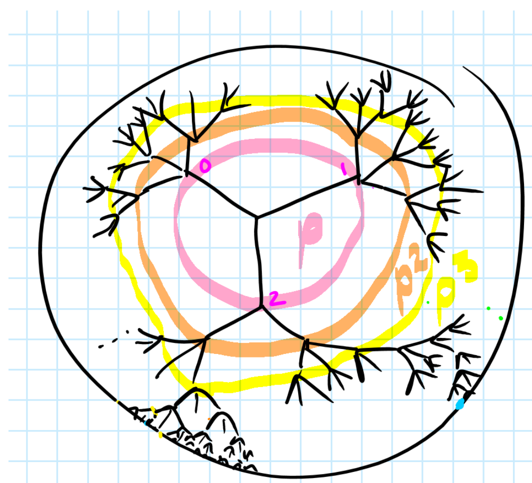
So, let's say that $B(c_1, r_1) \cap B(c_2, r_2)$ have non-trivial intersection, then, there exists $z \in B(c_1, r_1) \cap B(c_2, r_2)$. Then, $c_1 \sim_{r_1} z$ and $z \sim_{r_2} c_2$. Let's say $r_2 > r_1$. Then, $c_1 \sim_{r_2} z \sim_{r_2} c_2$. Let $y \in B(c_1, r_1)$, and $c_1 \sim_{r_1} y$. Then, $c_1 \sim_{r_2} y$ and $y \sim_{r_2} c_2$. Thus, $y \in B(c_2, y_2)$.

Thus, they are in the same equivalence class – if one ball contains the other than there is a totally ordered sequence of balls between them. This is a bit too abstract however.

2.3. **A Long Example to Help Think About The Balls in $\mathbb{Q}_p$.** Let us start by examining balls in $\mathbb{Z}_p$. We take the standard tree representation of $\mathbb{Z}_p$, where each path from the center of the tree to a point on the edge represents a string of digits.



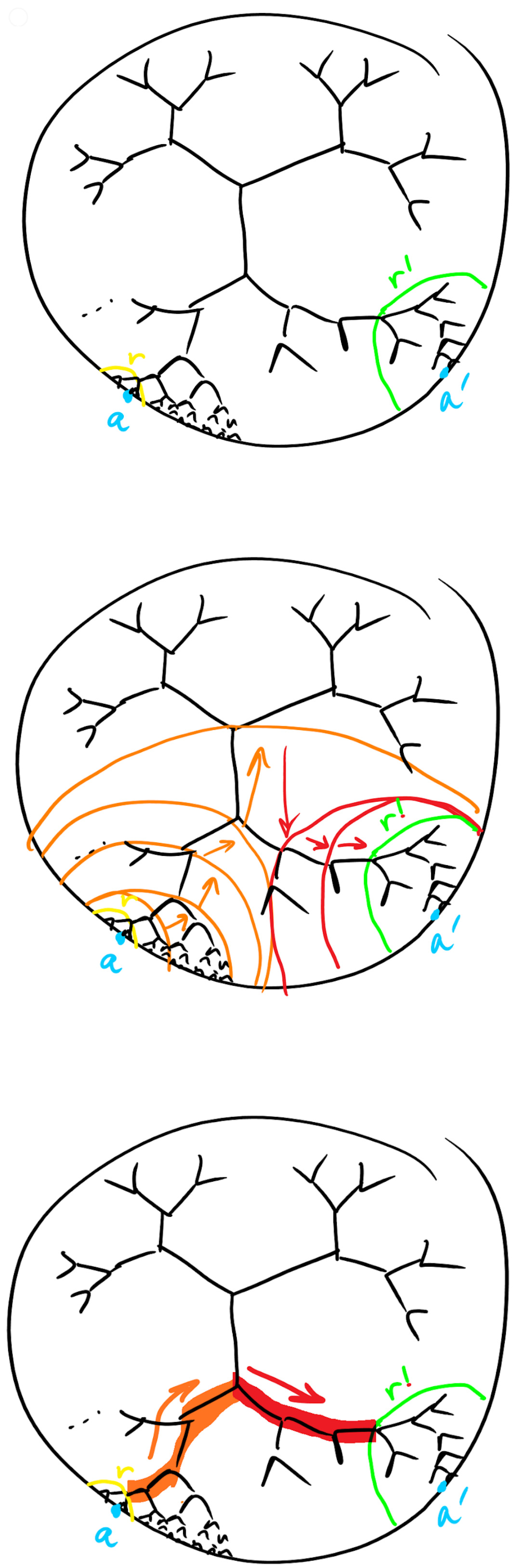
How does this digital representation correspond to the usual limit description?



There is a way to represent $\mathbb{Z}_p$ as a limit where the outer edge of the circle is $\mathbb{Z}_p$, and each radius of length k of the graph is the limit truncated at $\mathbb{Z}/p^k$.

We denote $\overline{x}$ as the truncated x, then, $\overline{x} = \overline{y}$ in $\mathbb{Z}/p^k$ iff $d(x, y) < p^{-k}$.

So, if we have $a, a' \in K$, considering them as 0 radius balls, the unique path of balls between them is as follows(same as more general case with r = 0). If we have $B(a, r), B(a', r') \subset K$, the unique path of balls between them is as follows:


r
r'
a
a'
r
r'
a
a'
r
r'
a
a'

Remark. WARNING: I messed up the previous 3 drawings but I don't feel like fixing it since it doesn't affect the path description. The following pictures should have valence 4 vertices away from the center.

Now, the path between $|\cdot|_{B(a,r)}$ and $|\cdot|_{B(a',r')}$ is the same in A^1_{Berk}. This nice tree picture is a lie in $\mathbb{Q}_p$. In fact, since $\mathbb{Q}_p \backslash \mathbb{Z}_p \simeq p\mathbb{Z}_p$ (this isomorphism takes $x \mapsto x^{-1}$ (flipping the ball of radius > 1 to the ball of radius < 1), we see that Q_p is just the $\mathbb{Z}_p$ tree with an extra "limb", making it an entirely equal valence graph. But – the ball shrinking and growing argument is the same.

What do your moving options look like as a point in this tree? There is only 1 path in to the Gauss point (which we will discuss later), and $\overline{\mathbb{F}}_p$ directions going out of any point.

2.4. **Other Norms on $K[T]$.** We can also make norms out of any decreasing nested sequence of closed discs, then the map is a multiplicative seminorm on $K[T]$:

$$f \mapsto \lim_{n\to\infty} |f|_{B(a_n,r_n)}$$

But why is it not just a ball norm, since the balls have discrete jumps? Because crazy bullshit exists: we can have a sequence of non-intersecting balls which accumulates to a point which is NOT in K. This is because K is just complete not "spherically" complete.

Question 12. What is an example of such a pathological sequence?

Theorem 9. *Berkovich then tells us that this is all of the possible multiplicative seminorms: Every point $x \in A^1_{Berk}$ corresponds to a nested sequence $B(a_1, r_1) \supseteq B(a_2, r_2) \supseteq \ldots$ of closed disks. Two nested sequences* (a) *iff* (a) *each has a nonempty intersection, and their intersections are the same, or* (b) *both have empty intersection, and the sequences are cofinal.*

Truly, there are three options of norms coming from three kinds of nested sequences:

- $\cap B(a_n, r_n) = B(a, r)$
- $\cap B(a_n, r_n) = a \in K$
- and, $\exists \cap B(a_n, r_n) = \emptyset$

Rephrasing this: there are 4 types of points in A^1_{Berk} according $B = \cap B(a_n, r_n)$.

(1) B is a point of K.
(2) B is a closed disk with radius belonging to $|K^*|$
(3) B is a closed disk with radius belonging to $|K^*|$
(4) $B = \emptyset$

There is a distinguished point: the Gauss norm. When we have a polynomial $f(T) = \sum_n a_n T^n$ in $\mathbb{Q}[T]$ and a prime p, we define the p-adic Gauss norm to be $|f|_p = \max_n |a_n|_p$. If $f(T) = c$ is constant, then $|f|_p = |c|_p$, so $|\cdot|_p$ on $\mathbb{Q}[T]$ restricts to the p-adic absolute value on $\mathbb{Q}$.

Question 13. How does the Gauss norm correspond to a point in $\mathbb{Q}_p$?

2.5. **Onto P^1_{Berk}.** As a set, the Berkovich projective lines P^1_{Berk} is obtained from A^1_{Berk} by adding a type 1 point at infinity.

Question 14. What is the point we add at infinity?

Remark. We should also define a sheaf of analytic function A^1_{Berk} and P^1_{Berk} and view them as locally ringed spaces endowed with extra structure of a maximal K-affinoid atlas.

Question 15. What is the sheaf of analytic functions?

This lecture was created with the help of [3] and [4].

3. Lecture 3: The Hodge-Tate Decomposition: an Introduction to Tools for Thinking about G_Q-representations

Well, I have been talking about the basic building blocks of 2 of the 3 p-adic categories. Grisha talked about adic rings and perfectoid spaces. So, I will move on to an application of perfectoid spaces.

So, what is this perfectoid stuff all about? One perspective is that it gives us a language with which we can speak about p-adic Hodge theory. First of all: what is p-adic Hodge theory? It seems to be the study if G_Q representations on p-adic vector spaces. We get a representation of G_Q or G_{Q_p} via getting an action of G_Q on a Q_p-vector space – for example, p-adic cohomology of a variety over Q.

Theorem 10. *Let $X/\mathbb{C}$ be an algebraic variety that is defined over $\mathbb{Q}$. Then, the absolute Galois group G_Q acts canonically on $H^i(X^{an}, Z/n)$ for any integer $n > 0$. Letting n vary through powers of a prime p, we obtain a continuous G_Q-action on the* $\mathbb{Z}_p$

Remark. Instead of just "adding" open sets to the Zariski topology to refine it, we consider not only open sets but also some schemes that lie over them. We consider all $\phi : U \to X$ which are etale. If X is a smooth variety over $\mathbb{C}$, this means that U is a disjoint union of smooth varieties, and ϕ is analytically an isomorphism.

Remark. For more on Galois representations in number theory
`https://mathoverflow.net/questions/103846/why-are-galois-representations-so-important-`

To understand these objects, one must first understand the action of the local Galois groups $D_\ell \subset G_Q$ (decomposition groups $= G_{Q_\ell}$?) at a rational prime ℓ. When $\ell \neq p$ (and the variety has good reduction), then these actions are classified by the action of a single endomorphism (the Frobenius) – and we can use the Weil conjectures to party in this circumstance. When $\ell = p$, the resulting representations are too rich to be understood in terms of a single endomorphism. Instead, these representations are best viewed as "p-adic Hodge structures."

What justifies this name? Well, we will try to build the result called "Hodge-Tate" decomposition starting from the usual Hodge decomposition. I should note that Hodge-Tate decomposition forms the first in a hierarchy of increasingly stronger statements describing the Galois representations on $H^n(X_{\overline{K},\acute{e}t}, \mathbb{Q}_p)$ in terms of the geometry of X.

3.1. **Hodge decomposition in $\mathbb{C}$.**

Theorem 11. *Let M be a compact Kähler manifold. Then, for all k, we have decomposition of $\mathbb{C}$-spaces:*

$$H^k_{sing}(M, \mathbb{C}) \simeq \bigoplus_{i+j=k} H^i(M, \Omega^j_{hol})$$

Morever, $h^{i,j}_M = h^{j,i}_M$ as $\mathbb{C}$-vector spaces.

Here, Ω^j_{hol} denotes the sheaf of holomorphic j-forms, and H^k_{sing} denotes the singular cohomology. The numbers $h^{i,j}_M$ denote $\dim_C H^i(M, \Omega^j_{hol})$ and are called Hodge numbers of M. Now, this beastie is proved by pretty analytic methods special to $\mathbb{C}$.

Remark. From this we can derive that for smooth projective integral curves, the algebraic and topological genus agree. The above theorem also tells us that the singular cohomology of the *manifold* X^{an} should have nothing to do with the sheaf cohomology on the scheme X, it does!!!

3.2. **Toward Hodge decomposition in $\mathbb{Q}_p$.** What should Hodge decomposition mean over $\mathbb{Q}_p$? Let's look back at the classical case. Namely, let's start with a projective, smooth, integral variety $X/\mathbb{Q}$. Then, let's write the Hodge decomposition like this:

Theorem 12. *("Theorem")*

$$H^k_{sing}(X^{an}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \bigoplus_{i+j=k} H^1(X, \Omega^j_{X/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

But – when you really think about it, taking the notion of "places" seriously – what is $\mathbb{C}$ but $\overline{\mathbb{Q}_\infty}$? So – what happens if we try and replace $\mathbb{C}$ with $\overline{\mathbb{Q}}_p$?

If we are going to try to replace $\mathbb{C}$ with $\overline{Q}_p$, what will we do with X^{an} and H^k_{sing}? We could use the rigid analytic analytification – which I alluded to in my first talk.

Remark. We may or may not talk about the process of analytification in this seminar, depending on time. I'm not sure what would happen if you used X^{an}, maybe a stronger related statement?

Even more simple mindedly, we can try replacing X^{an} with $X/\overline{\mathbb{Q}}_p$, and H^k_{sing} with some cohomology that acts like singular cohomology (that is, is a Weil cohomology theory). The only Weil cohomology on the wikipedia page which takes values in $\mathbb{Q}_p$ is p-adic cohomology.

Remark. A Weil cohomology is a cohomology satisfying certain axioms concerning the interplay of algebraic cycles and cohomology groups.

Remark. (Aside:) If anyone is uncomfortable with p-adic cohomology, we can come back to it.

$$H_{ét}(X; Z_p) := lim_n H_{ét}(X; Z/p^n)$$
$$H_{ét}(X; Q_p) := H_{ét}(X; Z_p) \otimes \mathbb{Q}$$

Étale cohomology works fine for coefficients Z/nZ for n co-prime to p, but gives unsatisfactory results for non-torsion coefficients

Question 16. Why does it give unsatisfactory results? $H^2_{ét}(P^1_C, \mathbb{Z}) = 0$ (BUT WHY– seems like Cech cohomology with cover $A^1 \cup A^1$ but it seems not???), by contrast, $H^2_{Betti}(P^1_C, \mathbb{Z}) = \mathbb{Z}$ as it is S^2.

So, when we make the formal replacements

$$\mathbb{R} \rightsquigarrow \mathbb{Q}_p \qquad \mathbb{C} \rightsquigarrow \overline{Q}_p \qquad X^{an} \rightsquigarrow X_{\overline{\mathbb{Q}}_p}$$

the Hodge decomposition theorem becomes

Theorem 13. *("Theorem")*

$$H^k(X_{\overline{\mathbb{Q}}_p, \mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \simeq \bigoplus_{i+j=k} H^i(X, \Omega_{X/\mathbb{Q}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_p$$

But we are in the land of Arithmetic, we expect more out of our isomorphisms. Namely, we don't expect them to just be isos of vector spaces – rather, they must respect their god given extra structure. Namely – surprise surprise the $G_{\mathbb{Q}_p}$ actions!

Namely, $G_{\mathbb{Q}_p}$ acts on the left of the above diagonally: $g(x \otimes y) = g(x) \otimes g(y)$, and on the right hand side, it acts only on the factor $\overline{\mathbb{Q}}_p$. But – it is not $G_{\mathbb{Q}_p}$ equivariant.

Why can there be no such isomorphism of $G_{\mathbb{Q}_p}$-modules? Well, let's forget about tensoring with $\overline{\mathbb{Q}}_p$. Is it possible that there is a $G_{\mathbb{Q}_p}$ isomorphism of the following form?

$$H^k(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq \bigoplus_{i+j=k} H^i(X, \Omega^j_{X/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$$

Nahhhh. Why? Simply because the left-hand side knows too much, whilst the right hand side doesn't! The right hand side is constant as a $G_{\mathbb{Q}_p}$ module. If this isomorphism was TRUE, then the left-hand side would have the same constant-ness! But, this cannot be!!!! For example, if X is an abelian variety over $\mathbb{Q}_p$, then the left-hand side as a $G_{\mathbb{Q}_p}$-module contains the information of whether or not X has good reduction at p. (It is a so called crystalline representation).

How do we fix this? It seems that if we hope for a $G_{\mathbb{Q}_p}$ module homomorphism, we must lose information. There is a natural way to do this:

Theorem 14. *Let K be a finite extension of $\mathbb{Q}_p$, and X/K be a smooth, integral, proper variety. Then, for all $k \geq 0$, we have a decomposition of G_K-modules:*

$$H^k(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=k} H^i(X; \Omega^j_{X/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}_p(-j)$$

Here, G_K, is still acting diagonally on the left, and just on the $\mathbb{C}_p(-j)$-factor on the right.

There are a few pieces of notation here:

- μ_{p^∞} is the p^nth roots of unity for all n.

- What does $\mathbb{C}_p(r)$ mean for r an integer? (let alone a rational lol). Well, we have a p-adic cyclotomic character $\chi_p : G_K \to Z_p^\times$ defined as the following composition:

$$G_K \to Gal(K(\mu_{p^\infty})/K) \simeq \mathbb{Z}_p^\times$$

We then denote by $K(1)$ the one-dimensional G_K-representation associated to χ_p.

Remark. Since the character is the trace of a representation, then, it is the representation itself when 1-dimensional.

Similarly, we denote by $K(r)$, for all $r \in \mathbb{N}$, the representation $K(1)^{\otimes r}$. Finally, we define $K(-r)$ for $r \in N$, $K(1)^\vee$, (the dual representation) with...I suppose...the dual G_K-action.

Then, for any K-linear G_K-module V and any $r \in \mathbb{Z}$, we denote by $V(r)$ the G_K-module $V \otimes_K K(r)$. Note that as a K-vector space, we can identify $V \otimes_K K(r)$ with V, so then the G_K action becomes

$$gx = \chi_p(g)^r gx$$

where gx is the G_K action on V we started with.

So on the right-hand side, the only non-trivial G_K-action exists on Tate twists, where it is defined as the tensor product of the G_K action on the two pieces. That is, $C_p(-j)$ is not a linear representation of G_K as a C_p-vector space; instead it is semilinear wrt the standard G_K action on $\mathbb{C}_p$.

3.3. **Perfection Itself.** There are two main steps in the approach

(1) Local study of Hodge cohomology via perfectoid spaces: Construct a pro-étale cover $X_\infty \to X$ which is "infinitely ramified in characteristic p", and study the cohomology of X_∞. (X_∞ should be an example of a perfectoid space). In particular, apparently, X_∞ gives no differential forms, so the full Hodge cohomology comes from the structure sheaf.
(2) Descent: Pull the Hodge cohomology of X_∞ down to X. In this step, we see that the differential forms on X, which vanished after our pullback to X_∞, reappear during our descent.

Remark. The pro-étale site is roughly the cite whose open subsets are roughly of the form $V \to U \to X$ where $U \to X$ is someétale morphism, and $V \to U$ is an inverse limit of finite étale maps. Then, the local structure of X in the pro-étale topology is locally perfectoid (this is basically the same as extracting lots of p-power roots of units in the tower $V \to U$).

We will in fact encounter the pro-étale site next lecture because we will construct universal covers of p-divisible groups (which are secretly their pro-étale covers).

Remark. The universal pro-etale cover is constructed like a usual universal cover: that is, it just has to satisfy the the universal property of a universal covering space (of covering maps factoring). Thanks to Jora for finding out why universal pro-etale covers exist, it is due to Lemma 7.18 in the paper on diamonds (https://arxiv.org/pdf/1709.07343.pdf).

I said here that next time we would talk about the proof of the abelian schemes case of the Hodge-Tate decomposition. However, instead I read a cool paper of Scholze and talked about that instead because p-divisible groups ESPECIALLY over nonfinite fields are my absolute weakness.

This was loosely based on [6] and [5].

4. Lecture 4: p-adic Analogue of Riemann's classification of Complex Abelian Varieties

First, I want to dissuade Jora of his feelings on étale cohomology of rigid-analytic varieties.

Theorem 15. *Let C be a completely algebraically closed extension of $\mathbb{Q}_p$, let X/C be a proper smooth rigid-analytic variety, let L be a $\mathbb{F}_p$-local system on X_{et}. Then, $H^i(X_{et}, L)$ is a finite dimensional $\mathbb{F}_p$-vector space for all $i \geq 0$, which vanishes for $i > 2\dim X$.*

The properness assumption is crucial here. However, using resolution of singularities, we can deduce the result for proper rigid-analytic varieties by resolution of singularities.

Let's talk about a result of Weinstein-Scholze on classifying p-divisible groups over $\mathcal{O}_C$, where C is an algebraically closed complete extension of $\mathbb{Q}_p$. We don't really need perfectoid stuff here. (Page 29 of [9]).

Why should we care? Well the weak Tate conjecture says (for a number field K):

Abelian Varieties of dimension g over $K \to$ Galois Representations $G_K \to GL_{2g}(\mathbb{Q}_p)$ has fibers consisting of isogeny classes.

Since Abelian varietyies are classified up to isogeny by their p-divisible group over K this is a GREAT motivation for understanding the classification of p-divisible groups over K!

4.1. The Classical Setting.

Theorem 16. *The category of complex tori is equivalent to the category of pairs $\mathfrak{t}, \Lambda$, where $\mathfrak{t}$ is a finite dimensional $\mathbb{C}$-vector space, and $\Lambda \subset \mathfrak{t}$ is a lattice.*

This classification can be reformulated in terms of Hodge Structures.

Definition 17.
- A $\mathbb{Z}$-Hodge structure of weight -1 is a finite free $\mathbb{Z}$-module Λ together with a $\mathbb{C}$-subvectorspace $V \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, such that $V \oplus V^* \simeq \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$.
- A polarization on a $\mathbb{Z}$-Hodge structure (Λ, V) of weight -1 is an alternating form
$$\phi : \Lambda \otimes \Lambda \to 2\pi i \mathbb{Z}$$
such that $\phi(x, Cy)$ is a symmetric positive definite form on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, where C is Weil's operator on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \simeq V \oplus V^*$, acting as i on V and as $-i$ on V^*.

So, it follows that complex tori are equivalent to $\mathbb{Z}$-Hodge structure of weight -1, mapping $(\sqcup, \Lambda)$ to (Λ, V), with $V = \ker(\Lambda \otimes \mathbb{C} \to \mathfrak{t})$.

Theorem 18. *The category of complex abelian varieties is equivalent to the category of polarizable $\mathbb{Z}$-Hodge Structures of weight -1.*

This theorem, stating an abstract equivalence between some geometric objects (abelian varieties) with some Hodge-theoretic data, the **"Hodge Theoretic perspective".**

In the case over $\mathbb{C}$, this equivalence has a very geometric meaning – all complex tori of dimension g have the same universal cover $\mathbb{C}^g$, and thus are of the form $\mathbb{C}^g/\Lambda$ for a lattice $\Lambda \subset \mathbb{C}^g$. When can one form the quotient $\mathbb{C}^g/\Lambda$? Always as a complex manifold, sometimes, as determined by Riemann, as an algebraic variety. We call this the "**geometric perspective**"

4.2. **p-adic analogue: The Hodge perspective.** Now, let C be an algebraically closed complete extension of $\mathbb{Q}_p$. Let X/C be a proper smooth scheme. (The following is true more generally for X/C a propert smooth rigid-analytic variety for which the Hodge-Tate spectral sequence degenerates.

If I felt I could communicate this without boring you to tears, I would start by telling you about The Hodge Tate Sequence for Abelian Varieties and p-divisible groups. But, I fear it is fairly technical. So, let me first motivate why we like it. This sequence gives us a functor from p-divisible groups over $\mathcal{O}_C$ to pairs (Λ, W), where Λ is a finite free $\mathbb{Z}_p$-module, and $W \subset \Lambda \otimes_{\mathbb{Z}_p} C$ is a C-subvectorspace.

Theorem 19. *This functor is an equivalence of categories.*

Sorry Grisha, here I will only tell you the functor, not why it is an equivalence of categories.

4.3. **What is this functor?** Let C be an algebraically closed complete extension of $\mathbb{Q}_p$.

We will take the example of p-divisible groups coming from abelian varieties to guide our way. This will show us what the functor is in all cases.

We will need to introduce the Hodge Tate and Hodge deRham spectral sequences. (see pg. 26 of [9])

These will give us two very important short exact sequences.

Let A/C be an abelian variety, with universal vector extension $EA \to A$, and p-adic Tate module Λ. Lie EA is apparently dual to $H^1_{dR}(A)$, and Λ is dual to $H^1_{\acute{e}t}(A, \mathbb{Z}_p)$.

Definition 20. What is dual abelian variety: we think of it as the connected component of the Picard scheme (the group of isomorphism classes of invertible sheaves (or line bundles) on X, with the group operation being tensor product)

Definition 21. Let A be an abelian variety over k. The universal vector extension of A is an extension EA of A by a vector group V, such that for each vector extension E of A there exist unique morphisms of algebraic groups f and F making the following diagram commute:

This says that E is the pushout of EA by $f : V_0 \to V$.

$$\begin{tikzcd} 0 \arrow[r] & V_0 \arrow[r] \arrow[d, "f"] & EA \arrow[r] \arrow[d, "F"] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Remark. Thanks to Jora for hunting down the defition of the universal vector extension EA: [http://www.martinorr.name/blog/2014/05/09/universal-vector-extensions/]

We get two short exact sequences, where A^* is the dual abelian variety.

$$0 \to (LieA^*)^* \to LieEA \to LieA \to 0$$

$$0 \to (LieA)(1) \to \Lambda \otimes_{\mathbb{Z}_p} C \to (LieA^*)^*$$

Oh, the second sequence is a little suggestive as a place to look for W don't you think? We must go deeper.

When we assume that A has good reduction, that is, $A/\mathcal{O}_C$ is also an abelian variety, where $\mathcal{O}_C$ is the ring of integers. Then, we can describe everything in terms of the p-divisible group $G = A[p^\infty]$. Indeed, we have the universal vector extension $EG \to G$, and the p-adic Tate module Λ of G. We have the short exact sequence of finite free $\mathcal{O}_C$ modules.

$$0 \to (LieG^*)^* \to LieEG \to LieG \to 0$$

Definition 22. (Cartier dual of p-divisible group) Let G = Spec A be a finite group over R, and $m : A \otimes A \to A$ define mult in A, and $\mu : A \to A \otimes A$ the group law. Let $A' = Hom_{R-mod}(A, R)$. Then, we get:

$$mu' : A' \otimes A' \to A, \qquad m' : A' \to A' \otimes A'$$

μ' defines an algebra structure on A' and m' defined a product on $G' = Spec(A')$. We call G' the cartier dual. But in this lecture we use the notation G^*.

Theorem 23. *(Fargues) There is a complex of finite free $\mathcal{O}_C$-modules,*

$$0 \to (LieG)(1) \xrightarrow{\alpha^*_{G^*}(1)} \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{\alpha_G} (LieG^*)^* \to 0$$

Its cohomology groups are killed by $p^{1/(p-1)}$

what does this mean, why is this killing important

This map α_G is constructed as follows: it matters because W will come from it's kernel!!! (see remark 4.14 pg 28 of [9])

HERE IS THE ALL IMPORTANT PROPOSITION. We already pretty much knew that the lattice Λ would be our friend the Tate module, but what the hell is W going to be? Well, we have two sequences. One we got from the Hodge-Tate-deRham spectral sequence, the second we get from Fargues's theorem.

Theorem 24. *Let $A/\mathcal{O}_C$ be an abelian variety, with Tate module Λ, and $G = A[p^\infty]$.*

$$0 \to LieA^* \otimes_{\mathcal{O}_C} C \to H^1_{ét}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to (LieA)^* \otimes_{\mathcal{O}_C} C(-1) \to 0$$
$$0 \to LieG \otimes_{\mathcal{O}_C} C(1) \to \Lambda \otimes_{\mathbb{Z}_p} C \to (LieG^*)^* \otimes_{\mathcal{O}_C} C \to 0$$

are dual to each other.

Remark. Remember, we can associate a Tate module (at p) to a p-divisible group – we can associate one to any damn group we please. Just take the limit as usual $T_p(G) := \lim G[p^n]$ where the maps are given by the multiplication by p-map $A[p^{n+1}] \to A[p^n]$

Remark. To be extremely precise about the definition of p-divisible groups, we are including $\{f \in \mathcal{O}(T) : f^{p^n} = 1\}$ into $\{f \in \mathcal{O}(T) : f^{p^{n+1}} = 1\}$

Question 17. What does Scholze mean by σ-linear algebra? (in the context of Deudonne Theory). He means our maps aren't linear – they have a factor σ – the lift of the Frobenius on $W(K) \to W(K)$.

Let K be the residue field of $\mathcal{O}_C$. Then, by reduction to the special fibre, one has a functor from p-divisible groups over $\mathcal{O}_C$ to p-divisible groups over K. Recall that the latter are classified by Dieudonne modules (M, F, V), where M is a finite free $W(K)$-module, $F : M \to M$ is a σ-linear map, and $V : M \to M$ is a $sigma^{-1}$-linear map, such that $FV = VF = p$.

So...now we have a functor from (Λ, W) to (M, F, V). Describing this functor amounts to an integral comparison between the etale and crystalline cohomology of p-divisible groups. WE WILL REVISIT THIS at the end of the talk because it is AWESOME.

Let K be a discretely valued complete extension of $\mathbb{Q}_p$, with perfect residue field. Then, HERE IS OUR FABULOUS HODGE CLASSIFICATION:

Theorem 25. *The category of p-divisible groups of $\mathcal{O}_K$ is equivalent to the category of lattices in crystalline representations of $Gal(\overline{K}/K)$ with Hodge-Tate weights 0, 1.*

Question 18. I think by crystalline representation we mean here $H^k(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$

4.4. **p-adic analogue: The Geometric perspective.** So, in this section, we learn for which $\mathbb{Z}_p$-modules Λ, we may quotient a p-divisible group by to get another p-divisible group.

Question 19. How can we think about a p-divisible group as an fpqc sheaf on schemes over R. We appear to need this to define the universal cover of G as $\tilde{G} = \lim_{\times p} G$. We end up with $ker(\tilde{G} \to G)$ as a sheafified version of the Tate module.

This functor turns $G \mapsto \tilde{G}$ isogenies into isomorphisms.

Question 20. Apparently the generic fiber of $\tilde{G}$ is perfectoid, but I still don't understand how being perfectoid helps at all with computation. Should I ask MO for examples?

Theorem 26. *Let G be a p-divisible group over $\mathcal{O}_C$. Then there is a p-divisible group unique up to isogeny $H/\overline{F}_p$, and a "quasi-isogeny"*

$$\rho : G \otimes_{\mathcal{O}_C} \mathcal{O}_C/p \to H \otimes_{\overline{F}_p} \mathcal{O}_C/p$$

Definition 27. Sheaves on the site of infinitesimal exntesions of open sets of X. Sheaces on this site grow – they can be extended from open sets to infinitesimal extensions of open sets. A crystal on this site $Inf(X/S)$ is a sheaf of $O_{X/S}$-modules which is rigid – any map f between objects T and T' of Inf(X/S) has the natural map $f^*F(T) \to F(T')$ be an iso.

So, in particular, we find that $\tilde{G} \simeq \tilde{H}_{\mathcal{O}_C}$. For any p divisible group G over $\mathcal{O}_C$, we have the $\mathbb{Z}_p$ lattice $TG(\mathcal{O}_C =: \Lambda$. Then, $\Lambda \subset \tilde{G}(\mathcal{O}_C)$. We get a fully faithful functor from (G, ρ), where ρ is as before, to the category of $\mathbb{Z}_p$-lattices $\Lambda \subset \tilde{H}(\mathcal{O}_C)$ Here, we use ρ to identify $\tilde{G}$ and $\tilde{H}$.

Thus, as in the case over $\mathbb{C}$, we can ask for which $\mathbb{Z}_p$ lattices $\Lambda \subset \tilde{H}(\mathcal{O}_C)$ one can form the quotient $\tilde{H}/\Lambda$ to get a p-divisible group.

Theorem 28. *The category of pairs (G, ρ) is equivalent to the category of $\mathbb{Z}_p$-lattices $\Lambda \subset H(\mathcal{O}_C$ such that the cokernel $V = coker(\Lambda \otimes C \to M(H) \otimes C)$ is of dimension d, and the sequence*

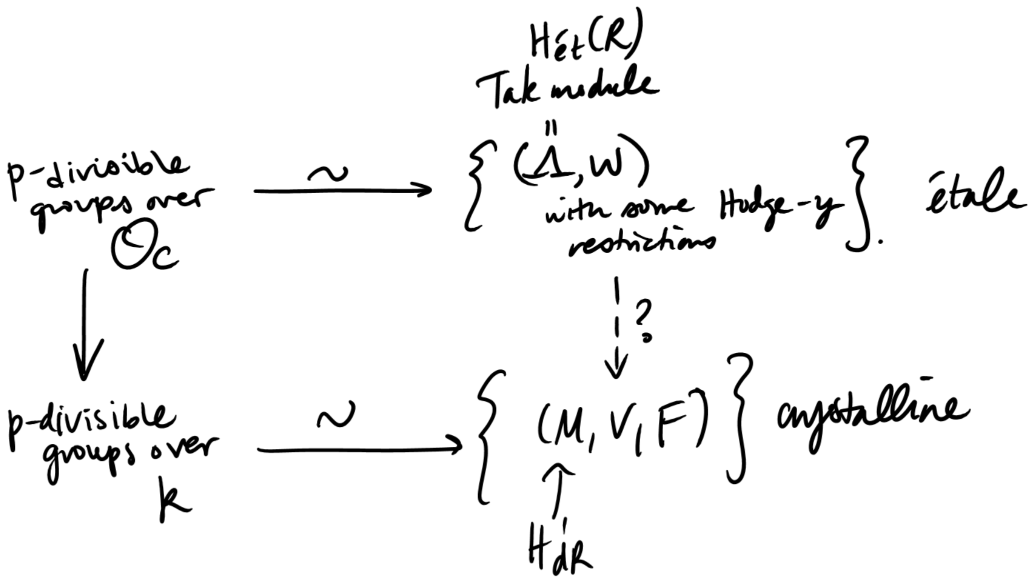
$$0 \to \Lambda[p^{-1}] \to H(\mathcal{O}_C) \to V \to 0$$

is exact

Remark. This map $\tilde{H}(\mathcal{O}_C) \to M(H) \otimes C$ is the quasi logarithm, where $M(H)$ is the covariant Dieudonne module.

Remark. This quasilogarithm is defined using the usual logarithm of p-divisible groups, $log_G : G(\mathcal{O}_C) \to LieG \otimes C$. To define the quasilogarithm properly we have to use classical Grothendieck-Messing theory – which relates deformations of p-divisible groups to lifts of the Hodge filtration (if the ideal defining the nilpotent immersion is equipped with a PD-structure).

Question 21*.* Thus we get a functor from p-divisible groups over K to p-divisible groups over $\mathcal{O}_K$, and thus from (Λ, W) to (M, V, F)? What is this functor?



Question 22*.* Does prismatic cohomology give a functor description between (Λ, W) and (M, V, F)? Maybe this is related to the Bellinson Fontaine conjecture?

Remark. We can think of having a mixed Hodge structure as being an action of Frobenius. The category of all mixed Hodge structures is Tannakian, that is, it is the representation category of some algebraic group. This algebraic group is called the Hodge-Galois group.

This lecture was based on [9].

5. Lecture 5: For The Love of Projective Space: Is Tilting Right? (What We Need for The Proof of the Monodromy Weight Conjecture)

In approaching the subject of p-adic geometry, due to its vastness, there are a few approaches. First to build a language or a foundation which is the "right one". To Grisha, this means the most universal definition which captures exactly the properties needed. To me, the "right definition" or "right conceptual landscape" is found by testing that one's definition has the properties you desire. Preferably, it can also be directly spoken to.

So, let's play. In the first two lectures we discussed A^1 in two adic categories. Now, we will discuss P^1.

Remark. This whole talk is based on [13], and I didn't type out most of my lecture here, so if you find these notes lacking, be sure to check those out. Further, feel free to skip this lecture and go straight to **Lecture 7** if you are willing to take the properties of projective spaces on faith.

5.1. **Is Tilting "Right"? How do P^n_K and $P^n_{K^\flat}$ talk to each other?** In all applications of perfectoid spaces, the hard part is to find a way to pass from objects of finite type over K to perfectoid objects. This is not possible in a canonical way, and one has to make a choice.

A perfectoid space is an adic space over K which is locally isomorphic to an affinoid perfectoid space, and the morphisms are morphisms of adic spaces.

Theorem 29.
$(P^n_K)^{perf}$ tilts to $(P^n_{K^\flat})^{perf}$

$$(P^n_K)^{perf} \sim \lim_{\phi} X^{ad}_K$$

Where ϕ is defined on coordinates by $[x_0 : \cdots : x_n] \mapsto [x_0^p : \cdots : x_n^p]$
There is a homeomorphism of both topological spaces, and more generally, etale topoi:

$$(X^{ad}_{K^\flat})^{\sim}_{\acute{e}t} \simeq \lim_{\phi} (X^{ad}_K)^{\sim}_{\acute{e}t}$$

If we take $U \subseteq |(P^n_K)|$, we have a commutative diagram of etale topoi:

$$\begin{tikzcd}
(\mathbb{P}^n_{K^\flat})^{ad,\sim}_{\acute{e}t} \arrow[r, "\pi"] & (\mathbb{P}^n_K)^{ad,\sim}_{\acute{e}t} \\
(\pi^{-1}(U))^{\sim}_{\acute{e}t} \arrow[r] \arrow[u, hook] & U^{\sim}_{\acute{e}t} \arrow[u, hook]
\end{tikzcd}$$

We did the proofs in the lecture on the fly just checking affine pieces, if you want to see more details, look at [13].

Remark. We will often build perfectoid spaces X/K from a filtered inverse system of Noetherian adic spaces

$$\phi_i : X \to X_i$$

We call them similar (we don't say equivalent because limits sometimes don't exist in adic spaces so yikies):

$$\lim_{\phi} X_i \sim X$$

if the following two conditions are satisfied:

(1) The maps on topological spaces $|X| \to |X_i|$ induce a homeomorphism $|X| \simeq \lim |X_i|$
(2) For any point $x \in X$, let x_i be the image in X_i, then we get maps $k(x_i) \to k(x)$ on the residue field. We require that the induced map $\lim k(x_i) \to k(x)$ has dense image.

Remark. Note that $k(x)$ is used to denote, I believe, *coker* : $|k|_x \to \mathbb{R}$, since each point in adic space is a valuation.

That is, the categories of presheaves on the "spaces" (where the topology is the etale topology) are equivalent.

Okay, what do all of these comparisons give us? Who cares?

Theorem 30. *Corollary:*

$$H^i((P^1_K)^{ad}_{\acute{e}t};\mathbb{Z}/p^n) \simeq H^i((P^1_{K^\flat})^{ad}_{\acute{e}t};\mathbb{Z}/p^n)$$

We must naturally ask, for what other adic spaces other than P^n does this theorem still hold, unchanged? Toric varieties, but we will not talk about them here, that is a whole 'nother talk.

5.2. **Pulling Back.** Try pulling back along these comparison theorems – everything BREAKS and is SO not a variety any more. Even just the example $x_0 + x_1 = 0$ becomes the infinite collection $\{x_0^{p^n} + x_1^{p^n}\}$ for all natural numbers n. That is quite NON algebraic!

So, we need to approximate here.

"geometrically connected proper smooth, set theoretic complete intersection with toric variety"

Remark. why do we need each of these conditions: the second condition comes from the proof of the approximation lemma

This lecture was largely based around [10] and [13]. Next lecture on the statement on the conjecture.

6. Lecture 6: Explaining the Weight-Mondodromy Conjecture

The monodromy weight conjecture is one of the main remaining open problems on Galois representations. It implies that the local Galois action on the ℓ-adic cohomology of a proper smooth variety is almost completely determined by the traces.

So, we will be looking at two different filtrations on $H^*_{\acute{e}t}(X_{\bar{K}};\overline{\mathbb{Q}}_\ell)$. They will come from the following exact sequence:

$$I_K \to G_K \to G_{\overline{\mathbb{F}}_p}$$

The monodromy filtration will come from I_K, which is the analogue of the first fundamental group in K.

The weight filtration will come from the weights of the eigenvalues of the geometric Frobenius on $G_{\overline{\mathbb{F}}_p}$.

6.1. Universal Properties of Both Filtrations.

Lemma 31. *(Simplified) Monodromy Filtration Given a finite dimensional vector space V with a nilpotent endomorphism N, there exists a unique increasing filtration $M_\bullet$ such that*

(1) $NM_i \subset M_{i-2}$
(2) $N^r : Gr_r^M V \simeq Gr_{-r}^M V$.

Lemma 32. *Monodromy Filtration Given a finite dimensional vector space V with a nilpotent endomorphism $N : V(1) \to V$, there exists a unique increasing filtration $M_\bullet$ such that*

(1) $M_{-k}V = 0$ and $MkV = V$ for sufficiently large k
(2) $N(M_k V(1)) \subset M_{k-1}V$ for all k
(3) $N^k : Gr_k^M V(k) \simeq Gr_{-k}^M V$

I want to emphasize here that this is pure linear algebra, when $N^2 = 0$ the filtration is simply $M_{-1} = \mathrm{im}(N)$, $M_0 = ker(N)$, $M_1 = V$.

Lemma 33. *Weight Filtration The unique increasing filtration such that*
$W_{-k}V = 0$ and $W_k V = 0$ for sufficiently large k
the action of I_K on Gr_k^W factors throat a finite quotient
after replacing K by a finite extension $Gal(F^{sep}/F)$ acts on $Gr_k^W V$ and this action has weight k.

We will discuss the existence of such filtrations later.

6.2. Monodromy in $\mathbb{C}$.

I simply use the first page of this for both Monodromy sections: [15]

6.3. Monodromy in K.

Let S be a Henselian DVR. Let K be $Frac(S)$, let k be its residue field of characteristic p. Let $\overline{K}$ be its algebraic closure. Let H be a finite dimensional vector space over $\mathbb{Q}_\ell$, $\ell \neq p$ on which G_K acts continuously.

Due to a theorem of Grothendieck,

$$I_\ell := \lim_n I_K/\ell^n I_K$$

is canonically isomorphic to $\mathbb{Z}_\ell(1)$.

$\mathbb{C}$	K
D	Spec(S)
$D*$	Spec (K)
universal cover $\tilde{D}^*$ of D^*	Spec($\overline{K}$)
$\pi_1(D^*)$	I_K
$\pi_(D^*) = \mathbb{Z} \simeq \mathbb{Z}(1)$	$I_\ell = \mathbb{Z}_\ell(1)$.
X	$X \to \mathrm{Spec}(S)$ projective scheme
$X^* = f^{-1}(D^*)$	X_K
$\tilde{X} := X \times_D \tilde{D}^*$	$X_{\overline{K}}$
$H^i(\tilde{X}; \mathbb{Z}_\ell)$	$H^i(X_{\overline{K}}, \mathbb{Z}_\ell)$

Thanks to Jora for discussing the local system piece with me, here is the original table of [16]. So on the left you have something over a punctured disk, and on the right you think of X/K as if it were a generic fiber of the integral model of X (that is, thinking of the special fiber as a "punctured disk"). You can prove theorems about the Galois action on cohomology similar to the ones from the usual monodromy theory.

9.1.

D	Spec (V)
D^*	Spec (K)
un revêtement universel $\tilde{D}^*$ de D^*	Spec $(\bar{K})$
groupe fondamental $\pi_1(D^*)$	groupe d'inertie I
(avec $\pi_1(D^*) = \mathbb{Z} \simeq \mathbb{Z}(1)_{\mathbb{Z}}$)	(avec $I_l = \mathbb{Z}_l(1)$)
X	schéma projectif X sur Spec (V)
$X^* = f^{-1}(D^*)$	X_K
$\tilde{X} = X \times_D \tilde{D}^*$	$X_{\bar{K}}$
système local $R^i f_* \mathbb{Z} \mid D^*$	module galoisien $H^i(X_{\bar{K}}, \mathbb{Z}_l)$
$H^i(\tilde{X}, \mathbb{Z})$	$H^i(X_{\bar{K}}, \mathbb{Z}_l)$

The pro-étale part of I_K has one generator (more on this next lecture), just like $\pi(D^*)$.

6.4. **Theorem that Logarithm of Monodromy is Nilpotent (The Monodromy Filtration).**

6.5. **Weight Spectral Sequence $\mathbb{C}$.** We start with a smooth quasi-projective complex variety U.

Thanks to Hironaka's resolution of singularities results, there exists a good "compactification" for any U, that is, an open embedding $U \subset X$ into a smooth projective variety such that $Y := X - U$ is a normal crossings divisor.

This is a formal sum of codimension d varieties, where etale locally, our intersection is transverse. More formally, we have smooth (projective) components $Y_1, .., Y_N$ such that each p-fold intersection of components $Y_{i_1} \cap ... \cap Y_{i_p}$ is smooth of pure codimension p (where $1 \leq i_1 < ... < i_p \leq N$).

Let $Y^{[q]}$ be the disjoint union of all q-fold intersections of the Y_i, more formally:

$$Y^{[}q] := \coprod_{i_1 < ... < i_q} Y_{i_1} \cap ... \cap Y_{i_q}$$

This gives us a combinatorial spectral sequence:

$$E_2^{p,q} = H^p(Y^{[q]}, \mathbb{Q}(-q) \Leftrightarrow H^{p+q}(U, \mathbb{Q})$$

Remark. $\mathbb{Q}(-q)$ is the tate twist $Q(2\pi i)$, that is, $ker(exp : \mathbb{C} \to \mathbb{C}^*)$.

We get the filtration which we use to build the spectral sequence from a filtration on the sheaf, by taking a sheaf on $X - U$ and pushing it onto U?

Our spectral sequence collpases at E_3. The differentials $d_r^{p,q}$ vanish for $r \geq 3$.

$$d_r^{p,q} : E_2^{p,q} \to E^{p+r, q-r+1}$$
$$H^p(Y^{[q]}, \mathbb{Q}(-q) \to H^{p+r}(Y^{[q-r+1]}, \mathbb{Q}(-q+r-1))$$

The left hand side is of weight $p - 2q$, and the right hand side is of weight $p + r + 2q - 2r + 2 = p + 2q - r + 2$.

Note the latter is not equal to the former when $r \geq 3$. Therefore, we can apply the following lemma:

Lemma 34. *If V and W are Hodge structures of weights $n \neq m$, then every morphism $\phi : V \to W$ of Hodge structures is trivial.*

A remarkable consequence is that the $E_3 = E_\infty$ term does not depend on the choice of compactification.

This section used [17]

6.6. **Weight Spectral Sequence K.** Let K be a fraction field of a Henselian DVR S.

Here is our translation table for this section:

$\mathbb{C}$	K
U	generic fiber $X \otimes_{\mathcal{O}_K} K$
$X - U =:$ Y	special fiber $X \otimes_{\mathcal{O}_K} F$
$Y^{[q]}$	$X^{(k)}$

Weights in characteristic p are much more intuitive. These are just coming from the eigenvalues of Frobenius. We will talk about this next lecture.

[16]

Remark. Keywords for the curious: Steenbrink on Mixed Hodge Structures & Formal Steenbrink Spectral Sequence

6.7. **Statement of the Conjecture.**

$$M_i = W_{i+n}$$

That is, the filtrations are equal up to a shift. We will discuss this shift in the next lecture.

6.8. **Equivalence of the Conjecture to a Statement in Terms of Spectral Sequence.** See Prop 2.5 and Remark 2.6 in [14]

Conjecture 35. *The monodromy operator $N^r : E_2^{-r,w+r} \simeq E_2^{r,w-r}$ for all r, w.*

The Weil conjectures can be thought of as: Let $X/\mathbb{Z}_p$ be a proper smooth curve.

$$H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \simeq H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$$

There is an action of $Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on the left, and of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the right. This iso is equivariant with respect to these.

The Monodromy-Weight Conjecture can be thought of as: Let $X/\mathbb{Z}_p$ be a sss. (more general).

Then, we can look at the reduction of $X/\mathbb{F}_p$, this looks like a bunch of intersecting lines. Each of these lines corresponds to: $H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)$. These lines being glued together gives us the differentials in our spectral sequence which collapses at the second page to be: $H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$.

So, we have a map:

$$H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \to H^*_{\acute{e}t}(X \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$$

which again respects the Galois action.

Note that the Weil conjectures are the special case for proper smooth varieties, when the monodromy action is trivial and the weight is automagically pure.

7. Lecture 7: On the Weight Filtration and Scholze's Proof for Hypersurfaces

7.1. **On the Notion of Weight.**

$$Z(V/\mathbb{F}_q;T) = \frac{P_1(T)\cdots P_{2N-1}(T)}{P_0(T)P_2(T)\cdots P_{2N}(T)}$$

where $P_i(T) \in \mathbb{Z}[T]$ with

$$P_0(T) = 1 - T \qquad P_{2N}(T) = 1 - q^N T$$

and such that for every $0 \leq i \leq 2N$, the polynomial $P_i(T)$ factors over $\mathbb{C}$ as

$$P_i(T) = \prod_{j=1}^{k_i}(1 - \alpha_{ij}T) \qquad \text{with } |\alpha_{ij}| = q^{1/2}$$

here, k is often called the ith Betti number of V.

The generalization of weight is fairly natural. Again, we have X a smooth variety over F_q, and $H = H^1(X_{\overline{\mathbb{F}_q}}; \overline{\mathbb{Q}_\ell})$. Then, the weight of an eigenvalue α of Frobenius is i if $|\alpha| = q^{i/2}$. Thus, one formulation of the weight monodromy conjecture is that all eigenvalues α of ϕ acting on $gr_j^M H^i$ are such that $|\alpha| = q^{(i+j)/2}$.

7.2. **Intro to Mixed Hodge Structures.**

The direct sum decomposition $H^m = \oplus_{p+q=m} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$ is a pure hodge structure of weight m.

The Hodge filtration

$$H^m = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

$$F^p = H^{p,m-p} \oplus H^{p+1,m-p-1} \oplus \cdots \oplus H^{m,0}$$

So F^p means at least p dz's. The spaces $H^{p,q}$ can be recovered $H^{p,q} = F^p \cap \overline{F^q}$.

Let's sat that X is a projective variety which is not necessarily smooth, or more grenerally, a quasi-projective variety (the difference of two such varieties). Deligne then tells us what the (p,q) type of a cohomology class should be. Let X be a quasi-projective variety. For each m, there is an increasing weight filtration

$$0 \subset W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H^m(X)$$

such that $gr_\ell = W_\ell / W_{\ell-1}$ for each ℓ has a pure Hodge structure of weight ℓ. In other words, gr_ℓ looks like the ℓth cohomology group of a smooth projective variety. [18]

7.3. **Restating the MW-Conjecture.**

Let $V^n := H^n(X_{\overline{K}}; \mathbb{Q}_\ell)$. Let $gr_i^M V = M_i V / M_{i-1} V$.

Conjecture 36.

$$gr_i^M V^n = W_{n+i}$$

7.4. **Deligne's Theorem.**

Theorem 37. *Let X be a curve, let (K be a finite field), $R := \hat{\mathcal{O}_{X,x}}$, $k = Frac(K)$. We look also at $k(X) \subset k$. Then, if the variety is defined over $k(X)$, the monodromy weight conjecture holds! The general case is for varieties defined over k, and not necessarily $k(X)$*

We fall further from God's grace everyday, from $\overline{K}$ to K to $K(\omega^{1/p^\infty})^\wedge$.

Remark. Note that Deligne here uses a replacement by a global field K, and Scholze uses a replacement by $k(\omega^{1/p_n})^\vee$. Why does moving from k to $k(\omega^{1/p^\infty}) =: K$ not affect the etale cohomology? It does not affect the Galois action by $G_{\mathbb{Q}_p}$ and it does not affect the maximal pro-ℓ quotient of the inertia group (looking at ω^{1/ℓ^n} where $\ell \neq p$) which is used in the monodromy filtration (see [9] pg 47 section 9 paragraph 1)

7.5. **Scholze's proof.**

You'll recall that we discussed two lectures ago **B**eautiful facts, that for any $U \subset P^n_K$, we have a commutative diagram of **E**tale topoi:

$$\begin{tikzcd} (\mathbb{P}^n_{K^\flat})^{ad,\sim}_{\acute{e}t} \arrow[r, "\pi"] & (\mathbb{P}^n_K)^{ad,\sim}_{\acute{e}t} \\ (\pi^{-1}(\mathcal{U}))^\sim_{\acute{e}t} \arrow[r] \arrow[u, hook] & \mathcal{U}^\sim_{\acute{e}t} \arrow[u, hook] \end{tikzcd}$$

Moreover, also discussed the sharp map:

$$K^\flat = \varprojlim_{x \mapsto x^p} K$$

$$K \to K \to K \to \cdots$$

$$x \mapsto x^p \mapsto x^{p^2} \mapsto \cdots$$

$$(x, x^p, x^{p^2}, \ldots)$$

$$\begin{aligned} K^\flat &\to K \\ (x, x^p, x^{p^2}, \ldots) &\mapsto x. \\ \| \quad & \quad\ \| \\ y \quad &\mapsto y^\# \end{aligned}$$

$$\mathbb{P}^n_{K^\flat} \xrightarrow{\pi} \mathbb{P}^n_K$$

$$[x_0 : \cdots : x_n] \longmapsto [x_0^\# : \cdots : x_n^\#]$$

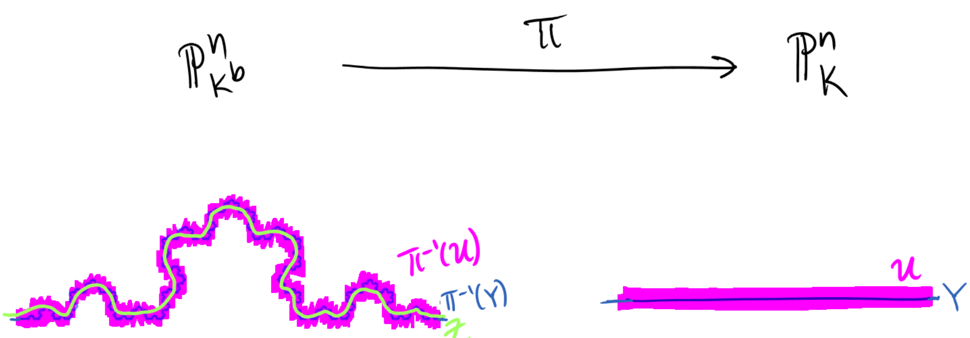
Adic world:

$$H^*(X_{\acute{e}t}; \mathbb{Q}_\ell) \simeq H^*(X^{ad}_{\acute{e}t}; \mathbb{Q}_\ell)$$

Doing the appropriate replacements:

$$\begin{tikzcd}
(\mathbb{P}^n_{K^\flat})^{ad,\sim}_{\acute{e}t} \arrow[r, "\pi"] & (\mathbb{P}^n_K)^{ad,\sim}_{\acute{e}t} \\
(\pi^{-1}(\mathcal{U}))^{\sim}_{\acute{e}t} \arrow[u, hook] \arrow[r] & (\mathcal{U})^{\sim}_{\acute{e}t} \arrow[u, hook] \\
(Z)^{ad,\sim}_{\acute{e}t} \arrow[u, hook] & (X)^{ad,\sim}_{\acute{e}t} \arrow[u, hook]
\end{tikzcd}$$

What is happening here, well, approximately (next to the above diagram):



Dimension of Z is equal to the dimension of Y. Let Z be a smooth proper **V**ariety contained in $\pi^{-1}(U)$. The above diagram is equivariant wrt the canonical Galois action of $G_K = G_{K^\flat}$ on this diagram such that all morphisms are G-equivariant.

So we get the G-equivariant map

$$H^i(Y_{\mathbb{C}_p,\acute{e}t}; \overline{\mathbb{Q}}_\ell) \to H^i(Z_{\mathbb{C}^\flat}; \overline{\mathbb{Q}}_\ell)$$

Theorem 38. ***I****so for* $i = 2\dim Y$

Proof. Here, $H^i(X)$ denotes $H^i(X; \overline{\mathbb{Q}}_\ell)$

$$\begin{tikzcd}
H^{2k}\left(\left(\mathbb{P}^n_{K^b}\right)^{ad,\sim}_{\acute{e}t}\right) \arrow[d] & H^{2k}\left(\left(\mathbb{P}^n_{K}\right)^{ad,\sim}_{\acute{e}t}\right) \arrow[l, "\sim"'] \arrow[d] \\
H^{2k}\left(\left(\pi^{-1}(\mathcal{U})\right)^{\sim}_{\acute{e}t}\right) \arrow[d] & H^{2k}\left(\mathcal{U}^{\sim}_{\acute{e}t}\right) \arrow[l] \arrow[d, "2"] \arrow[dl] \\
H^{2k}\left((Z)^{ad,\sim}_{\acute{e}t}\right) & H^{2k}\left((X)^{ad,\sim}_{\acute{e}t}\right)
\end{tikzcd}$$

If $H^{2k}(X) \to H^{2k}(Z)$ is not an isomorphism, it is the zero map, since this is a map on the cohomology of the top dimension.

This being the zero map implies that the restriction map $H^{2k}(P^n_{K^b}) \to H^{2k}(Z)$ is zero. But this cannot be. The kth power of the first chern class of the ample line bundle on P^n will have nonzero image in $H^{2k}(M)$.

Remark. The cohomology groups of P^n are generated by the class H of a hyperplane. If $f : X \to P^n$ is our embedding then by definition $f^*H = A$, an ample class. So, $f^*H^k = A^k$ must be nonzero.

□

From this, may deduce the MW-theorem.

Proof. Now, the Poincare duality pairing implies that $H^i(Z) \simeq H^i(Y) \oplus (otherjunk)$.

Remark. The poincare duality pairing is a map:

$$\begin{array}{ccc}
H^{2k-i}(Y) \oplus H^i(Y) & \xrightarrow{\mu_Y} & H^{2k}(Y) \\
{\scriptstyle f\times f}\big\downarrow & & \big\downarrow{\scriptstyle =} \\
H^{2k-i}(Z) \oplus H^i(Z) & \xrightarrow{\mu_Z} & H^{2k}(Z)
\end{array}$$

Since the pairing is nondegenerate, $H^i(Y) \xrightarrow{f} H^i(Z)$, if $ker(f) \neq 0$, then $im(f) \oplus coker(f) \simeq H^i(Z)$.

This is usually for homology, but we have rational coefficients so they are literally vector space dual.

By Deligne's theorem, $H^i(Z)$ satisfies the MW-conjecture, and thus, so does its direct summand $H^i(Y)$. □

Question 23. How limiting is considering only toric varieties?

This lecture is primarily based on the last bitty bits of [9].

8. Lecture 8: THE LAST LECTURE: On Dessin D'Enfant

This lecture was inspired by the series [19] and [20].

8.1. **On Maps.** Example of Monsieur Matthieu, map more general than dessin, make any map into a dessin.

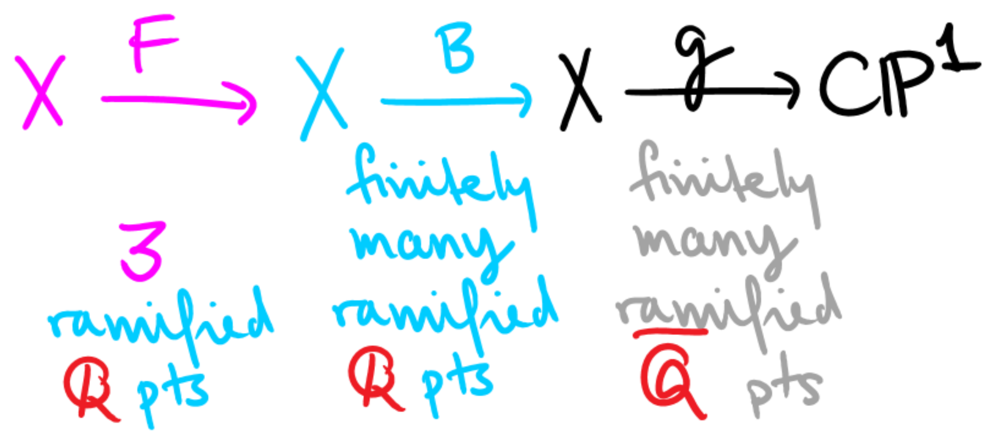
8.2. **Method of Labelling: Cartographic Groups.** Take the orientation imposed by the surface, start at a point and go around labelling what you see.

(open up a copy of sage)

```
K = PermutationGroup(['(1,2)(3,4)', '(1,3)(2,4)'])
K.order()
```

Equivalent maps give the same cartographic group: (example of Matthieu group). Define Matthieu group.

8.3. **Proof of Belyi's Theorem.** We can decompose this into two steps, first, given any X over $\overline{\mathbb{Q}}$,



8.4. **Grothendieck-Teichmüller Group.** We take the fundamental groupoid wrt the points which are maximally degenerate on $M_{g,n}$ (this is the moduli stack where we look at marked points with nontrivial tangent vector). Call this groupoid $T_{g,n}$. The maps from $M_{g,n-1} \leftarrow M_{g,n} \rightarrow M_{g,n+1}$ are forgetting and remembering points respectively. The maps between different genuses are cutting and gluing maps.

> There is another type of important morphism between the (orbifold) fundamental groups of the moduli spaces $M_{g,\nu} \rightarrow M_{g',\nu'}$ that is considered in Grothendieck's tower. You can see this morphism in three different ways. One is directly on the surfaces of type (g, ν) and (g', ν') (of genus g with ν boundary components, resp. genus g' with ν' boundary components). This morphism exists if you can put a set of disjoint simple closed loops on the surface of type (g', ν') such that when you cut along them, you cut your surface into one piece of type (g, ν), or else into several pieces of which at least one is of type (g, ν). You can also think of including the smaller surface of type (g, ν) into the bigger one by gluing it to other

smaller pieces along the edges of their boundary components, to form the bigger one of type (g', ν') (which is the image Grothendieck had in mind when he talked about Lego).

The second way to see this morphism is as a morphism of moduli spaces, where $M_{g,\nu}$ is mapped to a boundary component of the Deligne-Mumford compactification $\overline{M}_{g',\nu}$, in fact precisely the boundary component corresponding to taking the simple closed loops on the surface of type (g', ν') that "cut out" the one of type (g, ν) and shrinking them to length zero, so they become nodes.

The third way to view this same morphism is on the fundamental groups. This is pretty easy, since the (orbifold) fundamental group of $M_{g,\nu}$ is generated by Dehn twists along simple closed loops on the surface of type (g, ν), and these just map to the Dehn twists along the same simple closed loops when the (g, ν) surface is included in the (g', ν') one as above.

The Teichmller tower can be considered to be the collection of all the fundamental groups of the $M_{g,\nu}$ linked by the point-erasing morphisms and by these. Or, as Grothendieck wanted, instead of fundamental groups, that depend on a certain choice of base point, you can replace the groups by more symmetric fundamental groupoids based at all "tangential base points" on the moduli spaces".

The automorphism group of the Teichmller tower basically then consists of tuples $(\phi_{g,\nu})$ such that each $\phi_{g,\nu}$ is an automorphism of $\pi_1(M_{g,\nu})$ and the different $\phi_{g,\nu}$ in the same tuple commute with the homomorphisms of the tower.

– Schneps

Question 24. Why do we take it's profinite completion?

Thanks to Jora for asking me followup questions on this, and thanks to Nick Rosenblyum for conversing with me on the following (results of Fresse):

> the GrothendieckTeichmüller group, as defined by Drinfeld in quantum group theory, has a topological interpretation as a group of homotopy automorphisms associated to the little 2-disc operad

We may look at the GT group as $Aut(\cup_{n\in\mathbb{N}} M_{0,n})$, thinking of $M_{0,n}$ as a 2-disk operad. Note that $M_{0,n}$ is homotopy equivalent to the collection of $n-1$ little disks in the plane.

We may look at $M_{0,n} \to M_{0,k}$ as a composition of little disks, where we are gluing in the puncture at infinity to the original.

$M_{0,n} \times M_{0,m} \to M_{g,k}$, $g + k \simeq n + m$.

Question 25. I don't understand the reduction from a point in $M_{g,n}$ to $M_{0,k}$?

8.5. **Relation of G_Q and GT.** There are some constraints on the image of $G_{\mathbb{Q}}$ in GT, but the image is not yet understood. It is known that $G_{\mathbb{Q}} \subset GT$. Grothendieck conjectures that $GT \simeq G_{\mathbb{Q}}$.

8.6. **Examples of Dessin D'Enfant in Nature.** What's the dessin associated to $y^3 = x^4 - 1$? One of my favorite curves. It is a cylic 12-fold cover of the sphere ramified at 3 points (show why and the correponding tesselation).

9. The Talk Never Given: On the Nature of Good and Bad: Neron-Ogg-Shaferevich

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