AN INCOMPLETE PROOF THAT VARIETIES ARE FIBER BUNDLES OF FORMAL DISKS OVER THEIR DERHAM STACKS

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Let N be the nilradical.

Definition 1. We define the stack X_{dR} by its functor of points. $X_{dR}(R) := X(R/N)$.

Nick Rosenblyum mentioned the following theorem.

Theorem 2. *Everything is in char 0. Let D be the n-dimensional formal disk and let $\operatorname{Aut}(D)$ denote the group scheme of automorphisms of D.*

Then, if X is an n-dimensional smooth variety there is a canonical map

$$X_{dR} \to B\operatorname{Aut}(D)$$

such that the corresponding bundle associated to D is X. Here, X_{dR} is the deRham stack of X.

Remark. (Important) We here prove *only* that the fiber over any point of X_{dR} is a disk *when X is affine*. (We haven't done the etale descent argument yet). Thanks to Yaroslav Khromenkov for helping with the proof.

Note that in the etale topology, all X may be built out of affine pieces, $\mathbb{A}^n_R$. We prove first for X affine, then use etale descent.

Proof. We wish to prove that the pullback, Fiber of this diagram is a formal disk.

$$\begin{array}{ccc} \text{Fiber} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & X_{dR} \end{array}$$

As X_{dR} is only defined via its functor of points, we think of this diagram in terms of its functor of points.

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{Spec}\,R, \mathrm{Fiber}) & \longrightarrow & \mathrm{Hom}(\mathrm{Spec}\,R, \mathrm{Spec}\,A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathrm{Spec}\,R, \mathrm{Spec}\,k) & \longrightarrow & \mathrm{Hom}(\mathrm{Spec}\,R, X_{dR}) \end{array}$$

We rephrase the same diagram here:

$$\begin{array}{ccc} R(\mathrm{Fiber}) & \longrightarrow & \mathrm{Hom}(A, R) \\ \downarrow & & \downarrow \phi \\ \mathrm{Hom}(k, R) & \underset{\psi}{\longrightarrow} & \mathrm{Hom}(A, R/N) \end{array}$$

We note the definition of the pullback:

$$R(\mathrm{Fiber}) = \{(f,g)|\,\mathrm{Hom}(A,R) \times \mathrm{Hom}(k,R)|\phi(g) = \psi(f)\}$$

Understanding the meaning of the ϕ and ψ maps (let $\epsilon : \mathrm{Spec}\,k \to (\mathrm{Spec}\,A)_{dR}$, and $\epsilon^\sharp$ be the induced map in rings).

Rewriting this explicitly, $R(Fib)$ is all $(f,g) \in \mathrm{Hom}(A,R) \times \mathrm{Hom}(k,R)$ such that:

$$\begin{array}{ccc} k & \xrightarrow{g} & R \\ {\scriptstyle \epsilon^\sharp}\uparrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\phi(g)} & R/N \end{array} \quad = \quad \begin{array}{ccccc} & & R & & \\ & {\scriptstyle f}\nearrow & & \searrow{\scriptstyle g} & \\ A & & & & R/N \end{array}$$

In other words, $\phi(g) = \psi(f)$ means that the following diagram commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{\epsilon^\sharp} & k & \xrightarrow{g} & R & \xrightarrow{q} & R/N \\ & {\scriptstyle f}\searrow & & & & \nearrow{\scriptstyle q} & \\ & & & & R & & \end{array}$$

Note that g is unique because R is a k-algebra, so there is a unique map $g : k \to R$.

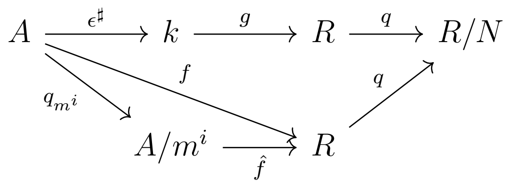
So,

$$R(\mathrm{Fiber}) = \{f \in \mathrm{Hom}(A,R)|\exists g \in \mathrm{Hom}(k,R) \text{ such that } \phi(g) = \psi(f)\}$$

Let $m := \ker(\epsilon^\#)$. Then, $\ker(f \circ q) = m$, and $\ker(f \circ q) = f^{-1}(N) \circ \ker(f)$. Note that

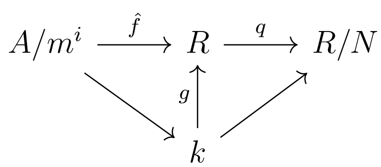
$$f^{-1}(N^i) \supseteq f^{-1}(N)^i$$

We require R to be Noetherian, which allows us to know that N is finitely generated. Since N is nilpotent and finitely generated, there exists a i such that $N^i = 0$, and thus $(f^{-1}(N))^i = 0$. Thus, for each f, the following commutes for some i:



We wish to show that a map f that factors through A/m^i is equivalent to a map f that satisfies $\phi(g) = \psi(f)$. We have shown how to go from $\phi(g) = \psi(f)$ to a map $A/m^i \to R$ (since q_{m^i} is the canonical quotient map). Thus, it remains to show that if we have a map $j : A/m^i \to R$, where $f := j \circ q_{m^i}$, then the diagram commutes. Note that $m := \ker(\epsilon^\sharp)$, and thus $k = A/m$, and since R and R/N are k-algebras, there is a unique map $k \to R$, and $k \to R/N$.

Note that m is nilpotent in m^i. Thus, under the map $q \circ \hat{f} : A/m^i \to R \to R/N$, m is sent to 0. Which means that this map factors through $A/m = k$. Now, the map $g : k \to R$ is unique because R is a k-algebra, thus the diagram commutes.



Thus, the condition of $\phi(g) = \psi(f)$ may be reformulated as follows:

$$R(\text{Fiber}) = \{h \in \text{Hom}(A/m^i, R), \forall i \; |A/m = k\}$$

Recall we are working in the category of prestacks, so the following holds (thanks Grigory Kondyrev):

$$\text{Spec}\, R \to \text{colim}_i \,\text{Spec}(A/m^i) = \text{colim}_i \,\text{Hom}(A/m^i, R) \Leftrightarrow A/m^i \to R \text{ for any } i$$

Thus, we see that by definition, $R(\text{Fiber}) = \text{Spf}(A, m)$. So, we have shown that a fiber over a point of the derham space of an affine scheme is a formal disk. $\square$