Overview of the Classic Theory of p-Divisible Groups

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We will discuss a victory of the 50s-60s: Lie theory for abelian schemes over perfect fields of char p. For this lecture, until mentioned otherwise, we will fix a field k which is perfect and of characteristic p. Further, everything is commutative.

1. Basic Definitons

Definition 1. Affine group scheme is $\operatorname{Spec} A$ where A is a bicommutative Hopf k-algebra.

Definition 2. Finite group scheme is an affine group scheme represented by a finite k-vector space A.

1.1. **$\underline{\mathbb{Z}/p^n}$, μ_{p^n}, α_p and their Hopf algebras.** For example:

- $\underline{\mathbb{Z}/p} = \operatorname{Spec} \operatorname{Hom}_{\mathsf{Sets}}(\mathbb{Z}/p, k) = \operatorname{Spec} \prod_{\mathbb{Z}/p} k = \coprod \operatorname{Spec} k$;
- $\mu_{p^n} = \operatorname{Spec} k[x]/(x^{p^n} - 1)$ is the pth roots of unity; $\Delta(x) = x \otimes x$.
- $\alpha_p = \operatorname{Spec} k[x]/x^p$ is the additive pth roots of unity; $\Delta(x) = x \otimes 1 + 1 \otimes x$.
- $E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p}$ (the kernel of multiplication by p on an ordinary elliptic curve)

1.2. **Definition and examples of p-divisible groups.** We give here examples of p-div groups, $\underline{\mathbb{Z}/p^n}$, μ_{p^n}, $A[p^n]$.

Definition 3. A **p-div group** of height h is an inductive system (i.e., an inductive limit before you take the limit) of commutative finite group schemes

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \cdots$$

over k satisfying two properties:

(1) They must fit into an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1} \to 0$$

(that is, the kernel of the map p^n on G_{n+1} is the copy of G_n sitting inside of G_{n+1}).

(2) $\operatorname{rank}(G_1) = p^h$ where h is an integer. (aka, $G_1 = \operatorname{Spec} A_1$, and A_1 is a free k-algebra of dimension p^n) This is called **height**.

Often, we will work with the individual finite group schemes rather than the whole colimit, as they are easier to handle.

2. Formal groups are connected p-divisible groups

Definition 4. An affine finite group scheme is **connected** if its representing Hopf algebra is a local ring.

Definition 5. A **divisible formal group** F is such that this sequence is exact (as a sequence of formal group schemes)

$$F[p] \to F \xrightarrow{[p]} F$$

Note that we may make a p-divisible group out of a divisible formal group scheme F.

$$F \mapsto (F[p] \hookrightarrow F[p^2] \hookrightarrow ...)$$

Remark. It is important here that our formal groups are indeed smooth formal groups, which implies that they locally look like Spf of a power series ring quotiented by a closed ideal.

Theorem 6. *(Serre-Tate equivalence) There is an equivalence of categories between divisible smooth formal groups over k, and connected p-divisible groups over k. The functor sends:*

$$F \mapsto (F[p] \hookrightarrow F[p^2] \hookrightarrow ...)$$

Let's look at some finite flat group schemes, and see what their connected components look like.

- $\operatorname{Spec} \mathbb{F}_p[x]/x^p$ is local, and thus connected.
- We see that for $\mu_{p/\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]/(x^p - 1)$ is local, since $(x-1)^p$, so $G^0 = G$, and thus $G/G^0 = \operatorname{Spec} \mathbb{F}_p$
- $\underline{\mathbb{Z}/p}$ is completely etale. (clearly disconnected)
- $E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p}$

Definition 7. An affine group scheme G over k is **etale** if $G \times_k \bar{k} \simeq \operatorname{Spec} \bar{k}[G(\bar{k})]$ (the coproduct of constant group schemes).

3. Connected-Etale Sequence and Splitting

Let G^0 be the connected component of the identity. Then, take $G^{ét} := G/G^0$.

$$0 \to G^0 \to G \to G^{ét} \to 0$$

This sequence in fact *always splits* (over a perfect field of char $p > 0$). And this splitting is natural!

We can think of this on the level of representing Hopf algebras, $A \simeq A_0 \otimes A_{ét}$.

Example 8. (analogy)

$$H^*(\Omega^\infty X; \mathbb{F}_p) \simeq \operatorname{Hom}(\pi_0 X, \mathbb{Z}/p) \otimes_{\mathbb{F}_p} H^*(\Omega^\infty X_0; \mathbb{F}_p)$$

(An example of a connected etale splitting of Hopf algebras, first part is "etale" (the only difference is that there could be an infinite number of connected components), second is connected)

Remark. Any map from G to an etale finite flat group scheme will factor through $G^{ét}$.

Remark. The fact that G^0 is a sub-group scheme relies on the general fact that if X is connected and has a rational point over the base field, then it is geometrically connected. If X is geometrically connected, then $X \times_k X$ is geometrically connected.

Remark. We define this exact sequence for finite flat group schemes, then take colimit to get the exact sequence for p-divisible groups.

4. Basics of Dieudonne Theory over k

4.1. **Dieudonne Theory I: Classification up to Isomorphism.** Moral: pDiv is equivalent to a category of modules, which one?

Where do these theorems come from? Manin proved them for a special case using the combination of a formal categorical statement of Gabriel, and some geometric input [2]. (Then he used descent and duality, discussed in the last section of [1], to prove the whole statement.)

We now discuss Gabriel's theorem on taking an abelian category and constructing a category of a modules.

4.1.1. *Gabriel's Theorem.*

Definition 9. An **injective hull** of an object c in abelian category $\mathcal{C}$ is a monomorphism $c \hookrightarrow I$ to an object $I \in \mathcal{C}$ such that:

- (injective) $\mathrm{Hom}(-, I)$ is exact
- (hull) $c \hookrightarrow I$ and there are no "smaller" I, that is, every other monomorphism from c to an injective object in $\mathcal{C}$ factors through this morphism.

Definition 10. Let a locally finite category be an abelian category with a finite set of generating objects, enough injectives, and "enough" colimits and limits.

Remark. (setup) Let C be a locally finite category, (S_α) the family of all simple objects of C, and I_α the injective hull of S_α. Let $I = \coprod I_\alpha$; the "universal" injective object. Let $E := \mathrm{End}_C(I)$. Topologize E by taking as a base of neighborhoods of zero the system of all left ideals $I \subset E$ of finite colength. E is complete wrt this topology. We denote by M_E the category of complete topological left E-modules, whose topology is linear and has a base of neighborhoods of zero consisting of all sub-modules of finite colength.

Theorem 11. *The contravariant functor $C \to M_E$, $X \mapsto \mathrm{Hom}(X, I)$ is an antiequivalence between the categories.*

There is little hope to compute injective objects for a general category, nor the endomorphisms of an object in a category. The geometric input, and how we get Dieudonne theory from this formal statement, is that we know the generators of the category of certain finite flat group schemes. This category we will consider if Ind of locally-local finite group schemes, $\mathsf{fGrp}^{\ell,\ell}$:

Definition 12. A finite group scheme $\mathrm{Spec}\, A$ is **locally-local**, if A and A^* are both local rings.

The generator of $\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})$ is α_p, and we know that the injective hull of α_p is the colim of truncated Witt-schemes.

Definition 13. The Witt scheme is a ring scheme whose k points are the rings of k-Witt vectors.

Remark. F and V act on a point of the Witt scheme as:

$$F : (x_0, x_1, x_2, ...) \mapsto (x_0^p, x_1^p, x_2^p, ...)$$
$$V : (x_0, x_1, x_2, ...) \mapsto (0, x_0, x_1, ...)$$

Definition 14.

$$W^r := \ker(W \xrightarrow{F^r} W)$$
$$W_s := \operatorname{coker}(W \xrightarrow{V^s} W)$$

$$W_s^r = \operatorname{Spec} k[x_0, ..., x_s]/(x_0^{p^r}, ..., x_s^{p^r})$$

colim W_s^r is an infinite dimensional formal group, an an object in $\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})$. Now all that is left is to understand

$$\operatorname{End}_{\mathsf{Ind}(\mathsf{fGrp}^{\ell,\ell})}(\operatorname{colim}_{r,s} W_s^r)$$

It ends up being

$$W(k)\{F, V\}/(\sim)$$

These are formal variables, let's discuss the equivalence relations. We need Cartier duality to think of these relations properly, but suffice to say (for $a \in W(k)$, where σ is the Frobenius in Witt vectors):

$$F(a) = \sigma(a)F$$
$$V(\sigma(a)) = aV$$
$$FV = p$$

Theorem 15. *(finite Dieudonne) There is a categorical anti-equivalence between finite group schemes of order p^h; and E-modules with $W(k)$-length n.*

Corollary 16. *(Dieudonne up to isomorphism) There is a categorical anti-equivalence between* pDiv *of height h; and free E-modules which are free as $W(k)$-modules (of rank h).*

Proof. Taking a colimit over diagrams on one side, and a limit on the other. The freeness comes from the fact that with torsion, the length of the $W(k)$-module doesn't grow fast enough to make it into the limit. ☐

4.2. **Dieudonne Theory II: Classification up to Isogeny.** Can we understand this category of modules? Well, to understand it we must make some sacrifices.

Definition 17. An isogeny is a map whose kernel is a finite flat group scheme.

Let E_F be $W(k)[\frac{1}{p}]\{F\}/(Fa = \sigma(a)F)$.

Theorem 18. *(Dieudonne up to isogeny) The category of p-divisible groups over up to isogeny $\mathsf{pDiv}^{\mathsf{isog}}$ has a fully faithful embedding into the category of finitely generated E_F-modules.*

Theorem 19. *(Dieudonne-Manin Classification Theorem)*
- *The category of finitely generated E_F-modules is semi-simple.*
- *If $k = \bar{k}$ the simple objects are of the form $G_{s,r} := E_F/E_F(F^r - p^s)$. This s/r is called the slope.*

Remark.
$$E_F/E_F(F^r - p^s) \simeq E_F \otimes_E E/E(F^{r-s} - V^s)$$

Remark. A technical point on simple objects in the case of general k. Given a simple module M over $\bar{k}$ of slope s/r, then $\mathrm{Aut}(M) = D^*$, where D is a skew field over $\mathbb{Q}_p$ with invariant s/r. There are different simple objects of slope s/r over k, they are of the form $H^1(\mathrm{Gal}(\bar{k}/k), D^*)$, for different actions of $\mathrm{Gal}(\bar{k}/k)$ on D^*.

Example 20. Let's discuss some isocrystals of familiar p-divisible groups:
- $\mu_{p^\infty} = \mathbb{G}_m[p^\infty]$ is isogenous to $G_{1/1}$
- $\underline{\mathbb{Z}/p^\infty} \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is isogenous to $G_{0/1}$
- For an ordinary elliptic curve, $G_{0/1} \oplus G_{1/1}$.
- For a supersingular elliptic curve, $G_{1/2}$
- For the Honda formal group of height h over $\mathbb{F}_p$ is $G_{1/h}$, $E_F/(F^h - p)$, this has basis $\{1, F, ..., F^{(h-1)}\}$ and
$$VF^i = \begin{cases} F^{h-1}, & i = 0; \\ pF^{i-1}, & 1 \le i \le h-1 \end{cases}$$

Remark. The left category $E_F{-}\mathsf{Mod}^{f.g.}$ is sometimes called the category of isocrystals.

5. p-divisible groups over a more general base

We can get pretty far with fields.

Theorem 21. *(Reynaud-Tate) (Tate's rigidity theorem) Let R be a local DVR with residue characteristic p, and $K := \mathrm{Frac}(R)$ of characteristic 0. Then, the generic fiber functor is fully faithful.*

$$\begin{aligned} \mathsf{pDiv}_{/R} &\to \mathsf{pDiv}_{/K} \\ G &\mapsto G_K \end{aligned}$$

This tells us that we can understand a p-divisible group over $\operatorname{Spec} R$ by its generic fiber. To do p-divisible groups over more general schemes, we repeat the original definition but make our constituent group schemes finite *flat*, rather than just finite. This was implicit before, but we were working over a field so everything was automatically flat.

Definition 22. A **p-div group of height h** is an inductive system (i.e., an inductive limit before you take the limit) of commutative flat group schemes (locally of finite presentation)

$$G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \cdots$$

over a base scheme S satisfying two properties:

(1) They must fit into an exact sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1} \to 0$$

(that is, the kernel of the map p^n on G_{n+1} is the copy of G_n sitting inside of G_{n+1}).

(2) $\operatorname{rank}(G_1) = p^h$ where h is a locally constant function $h : S \to \mathbb{Z}$

Fact: h is invariant under base change $S' \to S$. So p-divisible groups are great for deformation theory.

6. Serre-Tate Theorem

Set up. Let R be a base ring where p is nilpotent. $p \in I \subset R$ is a nilpotent ideal. Let $\mathsf{AbSch}_{/R}$ be the category of abelian schemes over R. Let $\mathsf{Def}(R, R_0)$ be the category of triples (A_0, G, ε) consisting of:

- $A_0 \in \mathsf{AbSch}_{/R_0}$
- $G \in \mathsf{pDiv}_R$
- an isomorphism in $\mathsf{pDiv}_{/R_0}$;

$$\varepsilon : G \times_R R_0 \xrightarrow{\simeq} A[p^\infty]$$

Theorem 23. *Serre Tate (Katz 1.2.1* [3]*): Let R and R_0 be as above. Then the functor*

$$\mathsf{AbSch}_{/R} \to \mathsf{Def}(R, R_0)$$
$$A \mapsto (A_0, A_0[p^\infty], \varepsilon)$$

is an equivalence of categories.

In other words, we can completely understand the deformations of an abelian scheme in terms of the deformations of its associated p divisible group.

Remark. Note that this does not depend on the choice of $G_0 := G \times_R R_0 \in \mathsf{pDiv}_{/R_0}$. That is, we are not fixing a group which we are lifting. This is a more general statement. We can take a fiber and recover the case that Betram and Alice discuss, where G_0 is fixed.

References

[1] R. Pink *Finite Group Schemes* ftp://ftp.math.ethz.ch/users/pink/FGS/CompleteNotes.pdf

[2] Yu. I. Manin *The theory of commutative formal groups over fields of finite characteristic* Russian Math. Surveys, 18:6 (1963), 1-83

[3] N. Katz *Serre-Tate Local Moduli* https://pdfs.semanticscholar.org/0c5c/37ff064e634eec9d239c4d1a3da3052d3aec.pdf