ZETA FUNCTIONS AND THH

KLEINES TOPOLOGIE SEMINAR JUNE 2025

Talk 3: Zeta Functions and THH (Rin Ray)

We are going to play with counting points on curves in finite characteristic.

Definition 0.1. *Given a smooth curve over $\mathbb{F}_q$, the zeta function is defined as*

$$\zeta(X,s) := \exp \sum_{m=1}^{\infty} \frac{|X(F_{q^m})|}{m} q^{-ms}.$$

Classically, the Grothendieck-Lefschetz formula (a la Weil conjectures), allows us to rewrite it.

Theorem 0.2. *Given a smooth curve over $\mathbb{F}_q$, the zeta function*

$$\zeta(X,s) = \prod_i \det(1 - \varphi q^{-s} | H^i(\overline{X}, \mathbb{Q}_\ell)^{(-1)^{i+1}}$$

Here φ denotes the action of Frobenius on the ℓ-adic cohomology of X.

Remark. Note that, if we so choose, we can make a convenient change of variables $Z(X, q^{-s}) = \zeta(X,s)$.

This is a meromorphic function, and we may define its holomorphic counterpart.

Definition 0.3.

$$\zeta^*(X,s) = \lim_{n \to s} (1 - p^{n-s})^{-\rho_n} \zeta(X,n),$$

where ρ_n is the order of the pole at $s = n$.

Remark. This is the first nonzero taylor series coefficient of $\zeta(X,s)$.

Main Goal. *Tease (Milne's Theorem (Mil86), Hyslop (Hys24)) Let n and s be integers, as $s \to n$ (p-adically)*

$$\zeta(X,s) \approx \chi(X, \mathbb{Z}_p^{\mathrm{syn}}(n), e) \chi(X, \mathcal{N}^{<n} W\Omega_X)(1 - p^{s-n})^{\rho_n},$$

where ρ_n denotes the order of the pole at $s = n$.

In other words, our main theorem can be stated as

Main Goal.

$$|\zeta^*(n)|_p = \chi(X, \mathbb{Z}_p^{\mathrm{syn}}(n), e) \chi(X, \mathcal{N}^{<n} W\Omega_X)(1 - p^{s-n})^{\rho_n}$$

It will turn out that the essence of this comparison is a lemma in linear algebra. We will show that the Grothendieck-Lefschetz form of the zeta function is comparable to the Lichtenbaum one.

Remark. These pieces on the right are the graded pieces of TC and its friend TC^+, defined later in the talk.

Before we show that, let's explore some context.

Date: Monday 14th July, 2025.

0.1. **Sidenote On Special Values for Number Fields and K-theory.** Why might you care about this strange creature ζ^*? For number fields, it's counter part is defined as follows.

Definition 0.4.

$$\zeta^*(\mathcal{O}_F, s) = \lim_{n \to s} (n + s - 1)^{\rho_s} \zeta(\mathcal{O}_F, s)$$

We define the regulator of Dirichlet, which computes the volume of a lattice.

Definition 0.5. *The Dirichlet regulator map is the logarithmic embedding,*

$$\rho_F^D \colon \mathcal{O}_F^\times / \mu_F \to \mathbb{R}^{r_1 + r_2 - 1}.$$

The covolume of the image lattice is the Dirichlet regulator R_F^D.

Remark. Recall that $r_1 + r_2 - 1$ is the rank of the units in $\mathcal{O}_F$.

Theorem 0.6. *(Dirichlet Class Number Formula)*

$$\zeta^*(\mathcal{O}_F, 0) = -\frac{\# \operatorname{Pic}(\mathcal{O}_F)}{\# \mu_F} R^{D_F}$$

We know that

$$K_0(\mathcal{O}_F) \simeq \mathbb{Z} \oplus \operatorname{Pic}(\mathcal{O}_F) \qquad K_1(\mathcal{O}_F) \simeq \mathcal{O}_F^\times.$$

Corollary 0.7. *(Dirichlet Class Number Formula)*

$$\zeta^*(\mathcal{O}_F, 0) = -\frac{\# K_0(\mathcal{O}_F)_{\text{tors}}}{\# K_1(\mathcal{O}_F)_{\text{tors}}} R_F^D$$

This is the historical start of class field theory and the first connection between zeta functions and K-theory, and what motivated the Lichtenbaum conjectures.

0.2. **ℓ-adic zeta functions and Tate twists.** Let's get back to linear algebra!

Theorem 0.8. *(Neu79) (3.1) Let X be a (geometrically connected algebraic $\mathbb{F}_q$-scheme). Let $\mathcal{F}$ be a $\mathbb{Z}_\ell$ sheaf on X_{et}, and $\mathcal{F}(n)$ its n-fold Tate twist, the for every integer n such that q^n is not an eigenvalue (i.e., no poles), then the cohomoloy groups are finite, trivial for $i > 2dim(X) + 1$, and*

$$|\zeta(X, n, \mathcal{F})|_\ell = \prod_i |H^i(X, \mathcal{F}(n))|^{(-1)^{i+1}}.$$

Proof. We use the linear algebra play below and the fact that by Lemma 0.9 (below) there's an isomorphism of Γ modules, $H^q(\overline{X}, \mathcal{F}(n)) \simeq H^q(\overline{X}, \mathcal{F})(n)$. This gives us that

$$(\varphi q^{-n} | H^i(\overline{X}, \mathcal{F}) \simeq (\varphi | H^i(\overline{X}, \mathcal{F}(n)),$$

Applying Lemma 0.9 we get

$$\begin{aligned} |\det(1 - \varphi q^{-n} | H^i(\overline{X}, \mathcal{F} \otimes \mathbb{Q}_\ell)|_\ell &= |\det(1 - \varphi | H^i(\overline{X}, \mathcal{F}(n) \otimes \mathbb{Q}_\ell)|_\ell \\ &= \prod_j |H^i(\Gamma, H^i(\overline{X}, \mathcal{F}(n))|^{(-1)^i} \end{aligned}$$

The following is a short exact sequence:

$$H^i(\overline{X},\mathbb{Z}_\ell(n))_\Gamma \to H^{i+1}(X,\mathbb{Z}_\ell(n)) \to H^{i+1}(\overline{X},\mathbb{Z}_\ell(n))^\Gamma$$

This gives us a lil spectral sequence

$$H^j(\Gamma, H^i(\overline{X},\mathcal{F}(n)) \Rightarrow H^{j+i}(X,\mathcal{F}(n))$$

$$\begin{aligned}
|\zeta(X,n,\mathcal{F})|_\ell &= \prod_i |\det(1-\varphi q^{-n}|H^i(\overline{X},\mathcal{F}\otimes\mathbb{Q}_\ell)|_\ell^{(-1)^{i+1}} \\
&= \prod_i\prod_j |H^j(\Gamma,H^i(\overline{X},\mathcal{F}(n))|_\ell^{(-1)^{i+j+1}} \\
&= \prod_i\prod_j |H^j(\Gamma,H^{i-j}(\overline{X},\mathcal{F}(n))|_\ell^{(-1)^{i+1}} \\
&= \prod_i\prod_j |H^j(X,\mathcal{F}(n))|_\ell^{(-1)^{i+1}}
\end{aligned}$$

We then smooth our spectral sequences from earlier to get back the formula we claim. □

Its this "Tate-twist" type interpretation of the zeta function that most directly connects to K-theory. (We talked about the relationship between etale cohomology and K-theory in the Tate-Poutou duality seminar.) Here is a sketch of the approach, again, it comes down to a linear algebra argument.

Lemma 0.9. *(Neu79) (3.2) Let $\Gamma = \mathrm{Gal}_{\mathbb{F}_q}$, and let φ be a topological generator of Γ. Let A be a finitely generated $\mathbb{Z}_\ell$-module with continuous Γ-action. (Then, if $det(1-\varphi|A\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)\neq 0$)*

$$|det(1-\varphi|A\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)|_\ell = \prod_i \#H^i(\Gamma,A)^{(-1)^i}.$$

Proof. (sketch)

$$\begin{aligned}
H^0(\Gamma,A) &= A^\Gamma = \ker(1-\varphi|A),\\
H^1(\Gamma,A) &= A_\Gamma = \mathrm{coker}(1-\varphi|A)
\end{aligned}$$

and $H^i = 0$ for $i>1$, sine Γ has cohomological dimension one. Consider the torsion submodule T of the $\mathbb{Z}_\ell$-module A. Since T is a finite Γ-module, the exact sequence

$$0\to T^\Gamma \to T \xrightarrow{1-\varphi} T_\Gamma \to 0,$$

shows that $|H^0(\Gamma,T)| = |H^1(\Gamma,T)|$. Therefore, we can assume that A is free. In this case A is a lattice in the finite dimensional vector space $A\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$. If $\det(1-\varphi)\neq 0$, then $H^0(\Gamma,A)\simeq 0$. We conclude by showing that $|H^1(\Gamma,A)| = |\det(1-\varphi|A\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)$ □

0.3. **p-adic zeta functions and syntomic cohomology.** That was a warmup for the equichar case. The definition of the zeta function is independent of choice of ℓ, and the following is also true:

Theorem 0.10. *For X a curve over a finite field,*

$$\zeta(X, s) = \prod_i \det(1 - \varphi q^{-s} | H^i(X, \mathbb{Q}_p)^{(-1)^{i+1}},$$

where we are taking the integral cristalline cohomology of X computed as the cohomology of the deRham-Witt complex $W\Omega_X^* \otimes \mathbb{Q}_p$.

Why might this be interesting? Well, we might want the p-local information (the p-valuation).

Let's define this. First, remember that the de Rham complex is $\Omega^0_{X/k} \to \Omega^1_{X/k} \to \cdots$, with differential given by d.

Theorem 0.11. *If X has a lift to $W(k)$, which we call $\overline{X}$, then, there is an iso*

$$H^*_{cris}(X) \simeq H^*_{dR}(\overline{X}).$$

Remark. Note that this is why we do not take the geometric completion of X in the equichar case!

If our X does not come with a lift to $W(k)$, (which it often doesn't, for example it doesn't if if X is supersingular), we cannot just compute the cristalline cohomology with said lift. Instead of lifting the geometric object X itself, we can just lift the algebraic de Rham complex Ω_X.

To define the deRham Witt complex, we must first define the Witt vectors.

Definition 0.12. *It's a function $W(-)\colon \mathrm{CRing} \to \mathrm{CRing}$ which sends $R \mapsto W(R)$. It's a ring with natural morphisms*

$$F : W(R) \to W(R) \qquad V : W(R) \to W(R),$$

such that $FV = p$. Also, the following sequence is short exact $0 \to W(R) \xrightarrow{V} W(R) \to R \to 0$.

Definition 0.13. *If k is an arbitrary $\mathbb{F}_p$-algebra, and X a scheme over k, there's a cdga $W\Omega^*_{X/k}$ called the deRham Witt complex, together with maps of graded groups*

$$F : W\Omega^*_{X/k} \to W\Omega^*_{X/k} \qquad V : W\Omega^*_{X/k} \to W\Omega^*_{X/k},$$

such that $FV = p$. Also, the following sequence is short exact $0 \to W\Omega^*X/k \xrightarrow{(V,dV)} W\Omega^*X/k \to \Omega^*_{X/k} \to 0$. If X is smooth and k, perfect, then

$$H^*_{cris}(X/W_n(k)) \simeq H^*(X, W_n\Omega^*_{X/k}),$$

where $W_n\Omega^*_{X/k} := W\Omega^*_{X/k}/(V^n, dV^n)$.

We will now rely again on a linear algebra formula (geometry of numbers feeling).

Lemma 0.14. *(Main Sauce!!) Suppose V is a finite dimensional $\mathbb{Q}_p$-vector space, and $F : V \to V$ is an automorphism of V. If there are $\mathbb{Z}_p$-lattices*

$$L' \subseteq L \subseteq V,$$

if F restricts to a linear map

$$F|_L^{L'} : L' \to L.$$

$$|\det(F)|_p = |\mathrm{coker}(F|_L^{L'} : L' \to L)|^{-1} \frac{|L|}{|L'|}.$$

Remark. Let's get some intuition here. Let a lattice Λ span R^n, and consider a Z-basis for the lattice, $\{a_1, ..., a_n\}$, and let A be a basis with those vectors as the rows. Consider R^n/Λ an n-dimensional torus, compact with finite volume equal to the fundamental domain $|detA|$. If Λ' is a sublattice of Λ, then

$$(R^n/\Lambda') = vol(R^n/\Lambda)|\Lambda/\Lambda'|$$

Our goal is to analyze the determinant of $1 - p^{-n}\varphi \colon H^*(X, \mathbb{Q}_p) \to H^*(X, \mathbb{Q}_p)$, we want to look for lattices relating to the rational crystalline cohomology of X. Natural choice is integral lattices, $W\Omega_X$ we just set up (thanks to Deligne-Illusie)

$$\mathcal{N}^{\leq n} W\Omega_X := (p^{n-1} VW\Omega_X \to p^{n-2} VW\Omega_X \to \cdots VW\Omega_X^{n-1} \to W\Omega_X^n \to \dots)$$

We have

$$\varphi p^{-n} : \mathcal{N}^{\leq n} W\Omega_X \to W\Omega_X,$$

and a map 1 induced by the filtration to $\mathcal{N}^{\geq 0} W\Omega_X \simeq W\Omega_X$,

Definition 0.15. *Recall that $\mathbb{Z}_p^{\mathrm{syn}}(n)(X) := (\varphi p^{-n} - 1 : \mathcal{N}^{\geq n} W\Omega_X \to W\Omega_X)$.*

Remark. When p is invertible on X, the syntomic complex coincides with the Tate twist

$$\mu_{p^n}^{\otimes i} \simeq (\mathbb{Z}/p^i)^{\mathrm{syn}}(n)(X).$$

Let's sketch the role of each object in the linear algebra theorem. In the following we mean $H^*(X, -)$ applied to each.

$$\begin{aligned}
F &:= (\varphi p^{-n} - 1) \colon W\Omega_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to W\Omega_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\
L' &:= \mathcal{N}^{\geq n} W\Omega_X \\
L &:= W\Omega_X \\
\mathrm{coker} F|_{L'}^{L} : L' \to L &:= \mathbb{Z}_p^{\mathrm{syn}}(n)(X) \\
L/L' &:= W\Omega_X / \mathcal{N}^{\geq n} W\Omega_X = \mathcal{N}^{<n} W\Omega_X.
\end{aligned}$$

We're all set up! From this, we check that our object play the correct roles, and thus prove our main goal.

Theorem 0.16. *(Milne's Theorem) Assume $X/\mathbb{F}_p$ is a smooth proper scheme of dimension d, and the Frobenius acts semisimply on $H^*(X, W\Omega_X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, then*

$$|\zeta^*(X,n)|_p^{-1} = \chi(X, \mathbb{Z}_p^{\mathrm{syn}}(n), e)\chi(X, \mathcal{N}^{<n} W\Omega_X).$$

in other words,

$$|\zeta^*(X,n)|_p^{-1} = \prod_i |(R^i\Gamma(gr_M^n TC(X)[-2n])R^i\Gamma(\pi_{2n}^{sh}(TC^+(X))))|_p^{(-1)^{i+1}}.$$

Proof. If there is no pole of $\zeta(X,s)$ at $s=n$, then the endormorphism $1-\varphi p^{-n}$ is invertible on each $H^i(X, W\Omega_X \otimes_{\mathbb{Z}} {}_p\mathbb{Q}_p$, then our lemma applies, because modulo torsion the lattices are contained in $H^*(X, W\Omega_X)$.

Let's look at the torsion, we have two long exact sequences of interest, coming from the cokernel vs the quotient.

$$\cdots H^i(X, \mathbb{Z}_p(n)) \to H^i(X, N^{\geq n} W\Omega_X) \xrightarrow{\varphi p^{-n}-1} H^i(X, W\Omega_X) \to \cdots$$

The torsion submodules of $H^i(X, W\Omega_X)$ and $H^i(X, N^{\geq n} W\Omega_X)$ contribute a factor of their quotient to the power $(-1)^{i+1}$, similarly, the long exact sequence coming from the quotient

$$\cdots H^i(X, \mathbb{Z}_p(n)) \to H^i(X, N^{\geq n} W\Omega_X) \to H^i(X, W\Omega_X) \to \cdots$$

contributes a factor of their quotients to the power $(-1)^i$. Thus, the torsion terms cancel in the product $\chi(X, \mathbb{Z}_p^{\mathrm{syn}}(n))\chi(X, \mathcal{N}^{\leq n} W\Omega_X)$.

$$|\det(F)|_p = |\mathrm{coker}(F|_L^{L'} : L' \to L)|^{-1} \frac{|L|}{|L'|}.$$

This implies that

$$\chi(X, \mathbb{Z}_p^{\mathrm{syn}}(n)) = \prod_i |\mathrm{coker}(F|_{L_i}^{L_i'} : L_i' \to L_i)|^{(-1)^{i+1}}$$

$$\chi(X, \mathcal{N}^{\leq n} W\Omega_X) = \prod_i |\frac{L_i}{L_i'}|^{(-1)^i}.$$

Thus, putting it all together, we get the conclusion. If there is a pole, this is where we need the assumption of semisimplicity: L_i and L_i' split up into $A_i \oplus B_i$ and $A_i' \oplus B_i'$ respectively, such that $B_i \to B_i'$ is an iso after inverting p. We focus on the A_i bit,

$$A_i' = \ker(F : L_i' \to L_i) \qquad A_i' = \ker(F : L_i' \to L_i).$$

We refer the reader to (Hys24) or (Sch82) for the rest of this argument. The essence is that we want to get a finite complex. To do this, it seems one shows we can forget about B and just work with A_i, and these turn out to correspond to cupping with an euler class. □

Remark. Away from poles of $\zeta(X,s)$,

$$C(X,n) = \zeta(X,n) = Z(X, p^{-n}).$$

$C(X,n)$ is the number we get after correcting for the pole at n (which is where this $(1p^{-n} - p^{-s})^{-\rho_n}$ term comes from, and is why we need to take a cup product with the Euler class e in that case to get a complex with finite cohomology).

We introduce a spectrum TC^+, analogously to TC^-, which is the fiber of the map $TC^+ := \mathrm{fib}(TC^- \xrightarrow{\text{can}} TP)$

$$gr^n_M TC(X)[-2n] \simeq \mathbb{Z}_p^{\text{syn}}(n)(A)$$
$$\pi^{sh}_{2n}(TC^+(X)) = N^{<n} W\Omega_X$$
$$\pi^{sh}_{2n}(TC^-(X)) = N^{\geq n} W\Omega_X$$

Remark. This follows from the picture Edith painted for us last time:

$$\begin{tikzcd}
R\Gamma_{syn}(\mathrm{gr}^n_M TC_A[-2n]) \arrow[r] \arrow[d, "\simeq"] & R\Gamma_{syn}(\pi^{sh}_{2n}(TC_A^-)) \arrow[r] \arrow[d, dashed, "\simeq"] & R\Gamma_{syn}(\pi^{sh}_{2n}(TP_A)) \arrow[d, "\simeq"] \\
\mathbb{Z}_p(n)(A) \arrow[r] & N^{\geq n}\Delta_A\{n\} \arrow[r] & \Delta_A\{n\}
\end{tikzcd}$$

Thus, our grand finale.

0.4. **End Remarks.** Why would this be interesting? Well, for example, this presentation of a zeta function in terms of localization invariants can be described and discussed more generally. We can also ask for an L-function which takes in number fields and function fields alike. Or even spectra (ongoing work of Gabe and me)!

Tate's thesis showed us that all L-functions can be presented adelically, and that this is their "correct form" in some sense, they can all be analytically continued. We can use that K-theoretic objects are described integrally to define correction terms/archimedian terms in the full L-function, as was done by Flach-Morin last year. This is not the case for TC, it is defined as a product over only nonarchimedian primes, we need an archimedian version of it as well. I believe this may be resolved by the ongoing foundational work of Wagner.

References

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