APPROACH TO RATIONAL CHROMATIC VANISHING

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Let J be the Morava Stabilizer group of a fixed h and p.

Lemma 1. $H^*(PGL_h(\mathbb{Z}_p), \mathbb{Z}_p) \otimes \mathbb{Q}_p \simeq H^*(J, W(k)) \otimes \mathbb{Q}_p$.

Let LT be the formal scheme associated to the Lubin-Tate ring. Let LT^η be the rigid generic fiber, and $LT^\infty := (LT^\eta)_\infty$ be the pro-etale cover of it. Let D be the Drinfeld upper half plane, and D^∞ be the Drinfeld tower.

Rational Chromatic Vanishing Conjecture.

$$H^*(J, W(k)) \otimes \mathbb{Q}_p \simeq H^*(J, LT) \otimes \mathbb{Q}_p.$$

Theorem 2. *To show the rational chromatic vanishing conjecture, it is sufficient to show that:*

(1) $H^*(J, LT) \otimes \mathbb{Q}_p \simeq H^*(J, LT^\eta)$

(2) $H^*(PGL_h, O_D) \simeq H^*(PGL_h, O_D^\eta) \simeq H^*(PGL_h, W(k))$.

Proof. Assume that (1) and (2) are true, here is a rough outline of how the rational vanishing conjecture follows. Here goes:

$$\begin{aligned}
H^*(J, LT) \otimes \mathbb{Q}_p \simeq H^*(J, LT^\eta) &= R\Gamma(LT^\eta // J) && (1)\\
&= R\Gamma((LT^\infty / GL_h(Z_p))//J) && (2)\\
&\simeq R\Gamma(LT^\infty // (J \times GL_h(\mathbb{Z}_p))) && (3)\\
&\simeq R\Gamma(D^\infty // (J \times GL_h(\mathbb{Z}_p))) && (4)\\
&\simeq R\Gamma(D // GL_h(\mathbb{Z}_p)) && (5)\\
&= H^*(PGL_h, O_D) && (6)\\
&\simeq H^*(PGL_h, W(k)) \otimes \mathbb{Q}_p && (7)\\
&\simeq H^*(J, W(k)) \otimes \mathbb{Q}_p && (8)
\end{aligned}$$

- (2) $\Rightarrow$ (3) is because $/ = //$ in the proetale category.
- (3) $\Rightarrow$ (4) is Fargue's two towers theorem (aka the geometric phrasing of Jacquet-Langlands).
- (7) $\Rightarrow$ (8) By lemma 1.

□

Remark. Regarding the choice in modding out by $GL_h(\mathbb{Z}_p)$ or $GL_h(Q_p)$ (similarly $J(\mathbb{Z}_p)$ or $J(Q_p) = D^\times$ for division algebra D), note that $LT^\infty/GL_h(\mathbb{Z}_p) = LT^\eta$, and $LT^\infty/GL_h(\mathbb{Q}_p) = \pi_{GH}(LT^\eta)$. Also note that $D^\infty/GL_h(\mathbb{Z}_p) = D$, and $D^\infty/GL_h(\mathbb{Q}_p) = HT(D)$ (hodge tate period map).

So, it remains to show:

Lemma 3. $H^*(J, LT) \otimes \mathbb{Q}_p \simeq H^*(J, LT^\eta)$

Approach: For (1) in Theorem 1, we will try to show an equivariant GAGA theorem.

Remark. For *p-adic* finite rings R (not LT), $(\mathrm{Spf}\, R)^\eta = \mathrm{Spa}\, R \otimes \mathbb{Q}_p$. It remains to show that we can cover $LT//J$ by p-adic finite rings.

Lemma 4. $H^*(PGL_h, O_D) \simeq H^*(PGL_h, O_D^\eta) \simeq H^*(PGL_h, W(k))$

Approach: We are trying to use a filtration of O_D given in a paper of Sasha Orlik to compute the cohomology of $H^*(PGL_h, F_i/F_{i+1})$. We were able to do this for O_D dual, but there was an issue that an admissible representation's dual wasn't admissible.