A Complete Proof of the Polynomial Ham Sandwich Theorem, Based on Gromov's Proof

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We take Gromov's proof idea from [1], which was extracted and summarized in section 1 of a paper by Guth [2]. We fill out some of the bits that Guth skimms over.

Claim: Given $U_1, ..., U_r$ finite volume subsets of $\mathbb{R}^n$, there exists a nontrivial degree d hypersurface which bisects each one, if $\binom{d+n}{n} - 1 \leq r$.

First, let's establish some terms:

Let RP^N be the space of all degree d hypersurfaces in n variables, here $N := \binom{d+n}{n} - 1$

Let $Bi(U)$ be the space of all degree d hypersurfaces in n variables which bisect the set U.

To say the polynomial P "bisects" $U \subset \mathbb{R}^n$, we mean that

$$\lambda(\{x \in U : P(x) \geq 0\}) - \lambda(\{x \in U : P(x) < 0\}) = 0$$

where λ is the standard Lesbesgue measure in $\mathbb{R}^n$.

Claim: $Bi(U)$ is nonempty.

Proof. Given a not null homotopic loop in RP^N, that is, $c \in \pi_1(RP^N)$. Let $c(0) = c(1) = P$, that is, P is a hypersurface of degree d. We may lift this loop to S^N.

$$\begin{tikzcd}
& S^N \arrow[d, "\pi"] \\
{[0,1]} \arrow[ur, dashed, "\tilde{c}"] \arrow[r, "c"] & RP^N
\end{tikzcd}$$

This loop may either remain a loop, that is, $\tilde{c}(0) = \tilde{c}(1)$, or, it may split, that is, $\tilde{c}(0) = -\tilde{c}(1)$. The lift must split because c corresponds to the nontrivial element of $\pi_1(RP^N) \simeq \mathbb{Z}/2$, which corresponds to the nontrivial deck transformation.

So, we have upstairs $\tilde{c}(0) =: P_0$, and $\tilde{c}(1) =: P_1 = -P_0$.

We define the function

$$F(t) := \lambda(\{x \in U : P_t(x) \geq 0\}) - \lambda(\{x \in U : P_t(x) < 0)\})$$

This function is well defined and can take any real number value for $P_t \in S_N$.

If $F(0) = 0$, then $\pi(P_0) \in Bi(U)$ so we are done. Assume that $F(0)$ is nonzero, then, $F(0) = k \in \mathbb{R}^\times$. Thus, $F(1) = -k$, because $P_1 = -P_0$, so the positive and negative components of F switch. By Guth's [2] *Continuity Lemma*, the function F varies continuously, so we may apply the intermediate value theorem: there must exist $\alpha \in (0,1)$ such that $F(\alpha) = 0$. Thus, $\pi(P_\alpha) \in Bi(U)$.

So, we have shown that every not null homotopic loop in RP^N contains an element of $Bi(U)$. Since $\pi_1(RP^N)$ is nontrivial, this establishes that $Bi(U)$ is nonempty.

□

We now wish to generalize this to show that $Bi(U_1, ..., U_r)$ is nonempty.

Claim: $Bi(U_1, ..., U_r)$ is nonempty.

Proof. We showed that every not null homotopic loop in RP^N is cut by $Bi(U)$. The contrapositive of this statement is that every loop in RP^N not cut by $Bi(U)$ is null homotopic in RP^N. We may restate this as the map:

$$\pi_1(RP^N - Bi(U)) \to \pi_1(RP^N)$$

is zero.

We may abelianize both sides using the Hurewicz map to get that the map on $\mathbb{Z}$-homology is also zero:

$$H_1(RP^N - Bi(U), \mathbb{Z}) \to H_1(RP^N, \mathbb{Z})$$

The functor $\mathrm{Hom}(-; \mathbb{Z}/2)$ takes the zero map to the zero map. Given a group map $f : A \to B$, the induced map on $f^* : \mathrm{Hom}(B, F) \to \mathrm{Hom}(A, F)$ takes $\beta \mapsto \beta \circ f$. Precomposing any group homomorphism with the zero map gives the zero map, so $f = 0 \implies f^* = 0$.

Now, we use the universal coefficient theorem, and the fact that H_0 is always free, to get that the isomorphism

$$H^1(X, \mathbb{Z}/2) \simeq \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}/2)$$

is natural and thus, the zero map is taken to the zero map. That is, the following map is zero

$$H^1(RP^N) \to H^1(RP^N - Bi(U_i))$$

Let a be the generator of $H^1(RP^N) \simeq \mathbb{Z}/2$. We then have this exact sequence:

$$\begin{array}{ccccc} H^1(RP^N, RP^N - Bi(U_i)) & \longrightarrow & H^1(RP^N) & \longrightarrow & H^1(RP^N - Bi(U_i)) \\ & & a & \longmapsto & 0 \end{array}$$

Since a is nonzero, this implies that there exists a_i in the preimage which is nonzero.

$$\begin{array}{ccccc} H^1(RP^N, RP^N - Bi(U_i)) & \longrightarrow & H^1(RP^N) & \longrightarrow & H^1(RP^N - Bi(U_i)) \\ \exists a_i & \longmapsto & a & \longmapsto & 0 \end{array}$$

By the definition of relative cup product, $\smile_{i=1}^r a_r \in H^r(RP^N, \bigcup_{i=1}^r RP^N - Bi(U_i))$. In the long exact sequence:

$$\begin{array}{ccccc} H^r(RP^N, \bigcup_{i=1}^r RP^N - Bi(U_i)) & \to & H^r(RP^N) & \to & H^r(\bigcup_{i=1}^r RP^N - Bi(U_i)) \\ \smile_{i=1}^r a_r & \longmapsto & a^{\smile r} & \longmapsto & 0 \end{array}$$

We assume that $r \leq N$, so that $H^r(RP^N) \simeq \mathbb{Z}/2 \neq 0$, which implies that $a^{\smile r}$ is nonzero. This implies that $\smile_{i=1}^r a_r$ is not zero.

Since $\smile_{i=1}^r a_r$ is a nonzero element of $H^r(RP^N, \bigcup_{i=1}^r RP^N - Bi(U_i))$, this implies that $H^r(RP^N, \bigcup_{i=1}^r RP^N - Bi(U_i)) \neq 0$.

Further,

$$\bigcup_{i=1}^r RP^N - Bi(U_i) = RP^N - \bigcap_{i=1}^r Bi(U_i) = RP^N - Bi(U_1, ..., U_r)$$

Thus, $H^r(RP^N, RP^N - Bi(U_1, ..., U_r)) \neq 0$, which implies $RP^N - Bi(U_1, ..., U_r) \neq RP^N$, which demonstrates that $Bi(U_1, ..., U_r)$ is nonempty.

□

We see that we crucially used the assumption that $r \leq N$. This is where the bound comes from!

References

[1] M. Gromov, *Isoperimetry of Waists and Concentration of Maps*. Geometric and Functional Analysis, Vol. 13, 2003.

[2] L. Guth, *The endpoint case of the Bennett-Carbery-Tao multilinear Kakeya conjecture.* https://arxiv.org/pdf/0811.2251v2.pdf